WEAKLY NULL SEQUENCES IN THE BANACH SPACE $C(K)$

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Abstract. The hierarchy of the block bases of transfinite normalized averages of a normalized Schauder basic sequence is introduced and a criterion is given for a normalized weakly null sequence in $C(K)$, the Banach space of scalar valued functions continuous on the compact metric space K, to admit a block basis of normalized averages equivalent to the unit vector basis of c_0, the Banach space of null scalar sequences. As an application of this criterion, it is shown that every normalized weakly null sequence in $C(K)$, for countable K, admits a block basis of normalized averages equivalent to the unit vector basis of c_0.

1. Introduction

We study normalized weakly null sequences in the spaces $C(K)$ where K is a compact metric space. When K is uncountable, $C(K)$ is isomorphic to $C([0,1])$ ([30], [34], [10]), while for every countable compact metric space K there exist unique countable ordinals α and β with $C(K)$ (linearly) isometric to $C(\alpha)$ [29] and isomorphic (i.e., linearly homeomorphic) to $C(\omega^{\omega^\beta})$ [13] (in the sequel, for an ordinal α we let $C(\alpha)$ denote $C([1,\alpha])$, the Banach space of scalar valued functions, continuous on the ordinal interval $[1,\alpha]$ endowed with the order topology).

Every normalized weakly null sequence (f_n) in $C(K)$ for countable K, admits a basic shrinking subsequence ([11], [15]) that is, a subsequence (f_{k_n}) which is a Schauder basis for its closed linear span and whose corresponding sequence of biorthogonal functionals is a Schauder basis for the dual of the closed linear subspace generated by (f_{k_n}).

It is shown in [28] that while (f_n) must admit an unconditional subsequence in $C(\omega^\omega)$, it need not admit an unconditional subsequence in $C(\omega^{\omega^2})$.

We remark here that if a normalized basic sequence in $C(K)$ for countable K has no weakly null subsequence, then it admits no unconditional subsequence since such a subsequence would have a further subsequence *equivalent* (this term is explained below) to the unit vector basis of ℓ_1 and $C(K)$ has dual isometric to ℓ_1 which is separable.

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Since $C(\alpha)$ is c_0-saturated for all ordinals α [35] (a Banach space is c_0-saturated provided all of its infinite-dimensional subspaces contain an isomorph of c_0), some *block basis* of (f_n) is equivalent to the unit vector basis of c_0.

We recall here that if (e_n) is a Schauder basic sequence in a Banach space then a non-zero sequence (u_n) is called a block basis of (e_n), if there exist finite sets (F_n), with $\max F_n < \min F_{n+1}$ for all n, and scalars (a_n) with $a_i \neq 0$ for all $i \in F_n$ and $n \in \mathbb{N}$ such that $u_n = \sum_{i \in F_n} a_i e_i$, for all $n \in \mathbb{N}$. We then call F_n the *support* of u_n. We shall adopt the notation $u_1 < u_2 < \ldots$ to indicate that (u_n) is a block basis of (e_n) such that $\max \operatorname{supp} u_n < \min \operatorname{supp} u_{n+1}$, for all $n \in \mathbb{N}$. We also recall that two basic sequences (x_n), (y_n) are equivalent provided the map T sending x_n to y_n for all $n \in \mathbb{N}$, extends to an isomorphism between the closed linear spans X and Y of (x_n) and (y_n), respectively. In the case T only extends to a bounded linear operator from X into Y, we say (x_n) *dominates* (y_n).

Our main results are presented mostly in Sections 3 and 6. We show in Corollary 6.8 that if (f_n) is normalized weakly null in $C(\omega^{\omega^\xi})$, one can always find c_0 as a block basis of *normalized α-averages* of (f_n) for some $\alpha \leq \xi$, and a quantified description of α is given. Note that the proof given in [35] of the fact that $C(\omega^{\omega^\xi})$ is c_0-saturated is an existential one that is, it only provides the existence of a block basis of (f_n) equivalent to the unit vector basis of c_0 without giving any information about the support of the blocks or the scalar coefficients involved. A normalized 1-average of $(f_m)_{m \in M}$ (where $M = (m_i)$ is an infinite subsequence of $\mathbb{N}$) is a vector $x = (\sum_{i=1}^{m_1} f_{m_i}) / \| \sum_{i=1}^{m_1} f_{m_i} \|$. Thus we have that the support of x is a maximal S_1-set in M where S_1 is the first Schreier class (see the definition of Schreier classes in the next section). A 2-average is similarly defined by averaging a block basis of 1-averages so that the support is a maximal S_2-set. This is carried out for all $\alpha < \omega_1$, as in the construction of the Schreier classes S_α, yielding the hierarchy of normalized α-averages of (f_n). The details are presented in Section 5.

Section 3 includes the following results. We show in Theorem 3.7 and Corollary 3.8 that if a normalized weakly null sequence (f_n) in $C(\omega^{\omega^\xi})$ is S_ξ-unconditional (see Definition 2.1 and the comments after it) then it admits an unconditional subsequence. This result, combined with that of [28] and [32] on Schreier unconditional sequences, yields an easier proof of the aforementioned fact about weakly null sequences in $C(\omega^\omega)$ [28]. Indeed, as is observed in [28] (see [32] for a proof), every normalized weakly null sequence in a Banach space admits, for every $\epsilon > 0$, a subsequence that is S_1-unconditional with constant $2 + \epsilon$. It follows from this and Theorem 3.7 that every normalized weakly null sequence in $C(\omega^\omega)$ admits an unconditional subsequence. Another consequence of Theorem 3.7 is that the example of a normalized weakly null sequence in $C(\omega^{\omega^2})$ without unconditional subsequence [28], fails to admit an S_2-unconditional subsequence

although of course it admits S_1-unconditional subsequences. This shows the optimality of the result in [28], [32] on Schreier unconditional sequences.

We show in Theorem 3.10 that if (χ_{G_n}) is a weakly null sequence of indicator functions in some space $C(K)$ then there exist $\xi < \omega_1$ and a subsequence of (χ_{G_n}) which is equivalent to a subsequence of the unit vector basis of the generalized Schreier space X^ξ ([1], [2]) (see Notation 3.3). We thus obtain a quantitative version of Rosenthal's unpublished result, that a weakly null sequence of indicator functions in some space $C(K)$ admits an unconditional subsequence (cf. also [8] and [7] for another proof of this result).

In Section 6 we give a sufficient condition for a normalized weakly null sequence in some $C(K)$ space to admit a block basis of normalized averages equivalent to the unit vector basis of c_0. We show in Theorem 6.1 that if (f_n) is normalized weakly null in $C(K)$ and there exist a summable sequence of positive scalars (ϵ_n) and a subsequence (f_{m_n}) of (f_n) satisfying $\{n \in \mathbb{N} : |f_{m_n}(t)| \geq \epsilon_{m_n}\}$ is finite for all $t \in K$, then there exist $\xi < \omega_1$ and a block basis of normalized ξ-averages of (f_n) which is equivalent to the unit vector basis of c_0. There are two consequences of Theorem 6.1. The first, Corollary 6.8, has been already discussed. The second one is Corollary 6.3, which gives a quantitative version of a special case of Elton's famous result on extremely weakly unconditionally convergent sequences [19] (cf. also [20], [22], [4] for related results). It was shown in [19] that if (x_n) is a normalized basic sequence in some Banach space and the series $\sum_n |x^*(x_n)|$ converges for every extreme point x^* in the ball of X^*, then some block basis of (x_n) is equivalent to the unit vector basis of c_0. We show in Corollary 6.3 that if (f_n) is a normalized basic sequence in some $C(K)$ space satisfying $\sum_n |f_n(t)|$ converges for all $t \in K$, then there exist $\xi < \omega_1$ and a block basis of normalized ξ-averages of (f_n) which is equivalent to the unit vector basis of c_0.

Finally, Sections 4 and 5 contain a number of technical results on α-averages which are used in Section 6.

Some of the results contained in this paper were obtained in B. Wahl's thesis [38] written under the supervision of E. Odell.

2. Preliminaries

We shall make use of standard Banach space facts and terminology as may be found in [27]. c_{00} is the vector space of the ultimately vanishing scalar sequences. If X is any set, we let $[X]^{<\infty}$ denote the set of its finite subsets, while $[X]$ stands for the set of all infinite subsets of X. If $M \in [\mathbb{N}]$, we shall adopt the convenient notation $M = (m_i)$ to denote the increasing enumeration of the elements of M.

A family $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ is *hereditary* if $G \in \mathcal{F}$ whenever $G \subset F$ and $F \in \mathcal{F}$. $\mathcal{F}$ is *spreading* if for every $\{m_1 <, \dots, < m_k\} \in \mathcal{F}$ and all choices $n_1 < \cdots < n_k$ in $\mathbb{N}$ with $m_i \leq n_i$ ($i \leq k$), we have that $\{n_1, \dots, n_k\} \in \mathcal{F}$. $\mathcal{F}$ is *compact*,

if it is compact with respect to the topology of pointwise convergence in $[\mathbb{N}]^{<\infty}$. $\mathcal{F}$ is *regular* if it possesses all three aforementioned properties and contains all singletons. A regular family $\mathcal{F}$ is said to be *stable*, provided that $F \in \mathcal{F}$ is a maximal, under inclusion, member of $\mathcal{F}$ if there exists $n > \max F$ with $F \cup \{n\} \notin \mathcal{F}$.

If E and F are finite subsets of $\mathbb{N}$, we write $E < F$ when $\max E < \min F$. Given families $\mathcal{F}_1$ and $\mathcal{F}_2$ whose elements are finite subsets of $\mathbb{N}$, we define their *convolution* to be the family

$$\begin{aligned}\mathcal{F}_2[\mathcal{F}_1] = &\{\cup_{i=1}^n G_i : n \in \mathbb{N},\, G_1 < \cdots < G_n,\, G_i \in \mathcal{F}_1 \,\forall\, i \leq n, \\ &(\min G_i)_{i=1}^n \in \mathcal{F}_2\} \cup \{\emptyset\}.\end{aligned}$$

It is not hard to see that $\mathcal{F}_2[\mathcal{F}_1]$ is regular (resp. stable), whenever each $\mathcal{F}_i$ is.

It turns out that for a regular family $\mathcal{F}$ there exists a countable ordinal ξ such that the ξ-th Cantor-Bendixson derivative $\mathcal{F}^{(\xi)}$ of $\mathcal{F}$ is equal to $\{\emptyset\}$. Hence $\mathcal{F}$ is homeomorphic to $[1, \omega^\xi]$, by the Mazurkiewicz-Sierpinski theorem [29]. We then say that $\mathcal{F}$ is of *order* ξ. If we define $\mathcal{F}^+ = \{F \in [\mathbb{N}]^{<\infty} : F \setminus \{\min F\} \in \mathcal{F}\}$, then it is not hard to see, using the Mazurkiewicz-Sierpinski theorem [29], that $\mathcal{F}^+$ is regular (and stable if $\mathcal{F}$ is) of order $\xi + 1$. It can be shown that if $\mathcal{F}_i$ is regular of order ξ_i, $i = 1, 2$, then $\mathcal{F}_2[\mathcal{F}_1]$ is of order $\xi_1 \xi_2$.

Notation. Given $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$ and $M \in [\mathbb{N}]$, we set $\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\infty}$. Clearly, $\mathcal{F}[M]$ is hereditary (resp. compact), if $\mathcal{F}$ is.

We shall now recall the transfinite definition of the Schreier families S_ξ, $\xi < \omega_1$. First, given a countable ordinal α we associate to it a sequence of successor ordinals, $(\alpha_n + 1)$, in the following manner: If α is a successor ordinal we let $\alpha_n = \alpha - 1$ for all n. In case α is a limit ordinal, we choose $(\alpha_n + 1)$ to be a strictly increasing sequence of ordinals tending to α.

Now set $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $S_1 = \{F \subset \mathbb{N} : |F| \leq \min F\} \cup \{\emptyset\}$. Note that $S_1 = S_1[S_0]$. Let $\xi < \omega_1$ and assume S_α has been defined for all $\alpha < \xi$. If ξ is a successor ordinal, say $\xi = \zeta + 1$, define

$$S_\xi = S_1[S_\zeta].$$

In the case ξ is a limit ordinal, let $(\xi_n + 1)$ be the sequence of successor ordinals associated to ξ. Set

$$S_\xi = \cup_n \{F \in S_{\xi_n + 1} : n \leq \min F\} \cup \{\emptyset\}.$$

It is shown in [1] that the Schreier family S_ξ is regular of order ω^ξ for all $\xi < \omega_1$. It is shown in [21] that the Schreier families are stable.

Definition 2.1 ([28], [32]). *A normalized basic sequence (x_n) in a Banach space is said to be Schreier unconditional, if there exists a constant $C > 0$ such that $\|\sum_{n \in F} a_n x_n\| \leq C \|\sum_n a_n x_n\|$, for every $F \subset \mathbb{N}$ with $|F| \leq \min F$, and all choices of finitely supported scalar sequences (a_n).*

It has been already mentioned in the introductory section that every normalized weakly null sequence admits, for every $\epsilon > 0$, a subsequence that is Schreier unconditional with constant $2 + \epsilon$.

The concept of Schreier unconditionality can be generalized in the following manner: Consider a hereditary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ containing the singletons. A normalized basic sequence (x_n) is now called *$\mathcal{F}$-unconditional*, if there exists a constant $C > 0$ such that $\|\sum_{n\in F} a_n x_n\| \leq C\|\sum_n a_n x_n\|$, for every $F \in \mathcal{F}$ and all choices of finitely supported scalar sequences (a_n).

3. Upper Schreier estimates

In this section we show that every normalized weakly null sequence in $C(K)$, K a countable compact metric space, admits a subsequence dominated by a subsequence of the unit vector basis of a certain Schreier space (see the relevant definition after the statement of Theorem 3.9).

Recall, [13], that for every countable compact metric space K, there exists a unique countable ordinal α with $C(K)$ isomorphic to $C(\omega^{\omega^\alpha})$. Since most of the properties of weakly null sequences in $C(K)$ that we shall be interested in, are isomorphic invariants, there will be no loss of generality in assuming that $K = [1, \omega^\xi]$, for some $\xi < \omega_1$. As is has been already mentioned in the previous section, every regular family $\mathcal{F}$ of order ξ (this means $\mathcal{F}^{(\xi)} = \{\emptyset\}$) is homeomorphic to the ordinal interval $[1, \omega^\xi]$. Moreover, it is easy to construct by transfinite induction, a regular family of order ξ, for all $\xi < \omega_1$. We can thus identify $C(\omega^\xi)$ with $C(\mathcal{F})$, for every regular family of order ξ.

The advantage of such a representation is that one can easily construct a monotone, shrinking Schauder basis of $C(\mathcal{F})$, the so-called *node basis* [3]. Indeed, let $(\alpha_n)_{n=1}^\infty$ be an enumeration of the elements of $\mathcal{F}$, compatible with the natural partial ordering of $\mathcal{F}$ given by initial segment inclusion. This means that whenever α_m is a proper initial segment of α_n, then $m < n$. In particular, $\alpha_1 = \emptyset$. Such an enumeration is for instance, the anti-lexicographic enumeration of the elements of $\mathcal{F}$, i.e., $F \prec G$ if and only if either $\max F < \max G$, or $F \setminus \{\max F\} \prec G \setminus \{\max G\}$, for all F, G in $\mathcal{F}$.

Given $\alpha \in \mathcal{F}$, set $G_\alpha = \{\beta \in \mathcal{F} : \alpha \leqslant \beta\}$, where $\alpha \leqslant \beta$ means that α is an initial segment of β. Clearly, G_α is a clopen subset of $\mathcal{F}$ for every $\alpha \in \mathcal{F}$. The sequence $(\chi_{G_{\alpha_n}})_{n=1}^\infty$ is called the node basis of $C(\mathcal{F})$. It is not hard to check that $(\chi_{G_{\alpha_n}})_{n=1}^\infty$ is a normalized, monotone, shrinking Schauder basis for $C(\mathcal{F})$ [3].

Proposition 3.1. *Let $\mathcal{F}$ be a regular family and $u_1 < u_2 < \dots$ be a block basis of the node basis $(\chi_{G_{\alpha_n}})_{n=1}^\infty$ of $C(\mathcal{F})$. Then there exist positive integers $n_1 < n_2 < \dots$ with the following property: For every $\gamma \in \mathcal{F}$, $\{n_i : i \in \mathbb{N},\, u_{n_i}(\gamma) \neq 0\} \in \mathcal{F}^+$.*

Proof. Define $F_n = \{\alpha_i : i \in \operatorname{supp} u_n\}$, for all $n \in \mathbb{N}$. Clearly, the F_n's are pairwise disjoint, finite subsets of $\mathcal{F}$. We observe that whenever $\alpha_i \in F_n$

and $\alpha_j \in F_m$ satisfy $\alpha_i \leqslant \alpha_j$, then $n \leq m$. This is so since $\alpha_i \leqslant \alpha_j$ implies that $i \leq j$ and, subsequently, that $u_n \leq u_m$. Hence, $n \leq m$.

We next choose inductively, integers $2 = n_1 < n_2 < \dots$ such that $\max \beta < n_{i+1}$ for every $\beta \in F_{n_i}$ and all $i \in \mathbb{N}$ (where, $\max \beta$ denotes the largest element of the finite subset β of $\mathbb{N}$). We claim (n_i) is as desired. Indeed, let $\gamma \in \mathcal{F}$. Then

$$\{n_i : i \in \mathbb{N},\, u_{n_i}(\gamma) \neq 0\} \subset \{n_i : i \in \mathbb{N},\, \exists\, \beta \in F_{n_i},\, \beta \leqslant \gamma\},$$

for writing $u_{n_i} = \sum_{\beta \in F_{n_i}} \lambda_\beta \chi_{G_\beta}$ for some suitable choice of scalars $(\lambda_\beta)_{\beta \in F_{n_i}}$, we see that $u_{n_i}(\gamma) \neq 0$ implies $\chi_{G_\beta}(\gamma) = 1$, for some $\beta \in F_{n_i}$ with $\beta \leqslant \gamma$. In particular, $\{n_i : i \in \mathbb{N},\, u_{n_i}(\gamma) \neq 0\}$ is finite. Let now $\{n_{i_1} <, \dots, < n_{i_k}\}$ be an enumeration of $\{n_i : i \in \mathbb{N},\, u_{n_i}(\gamma) \neq 0\}$, and choose $\beta_j \in F_{n_{i_j}}$ with $\beta_j \leqslant \gamma$, for all $j \leq k$. Since $\{\beta_1, \dots, \beta_k\}$ is well-ordered with respect to the partial ordering $\leqslant$ of $\mathcal{F}$ (all the β_j's are initial segments of γ), our preliminary observation yields $\beta_1 < \dots < \beta_k$. Note that $\beta_1 \neq \emptyset$. By the choices made, $\max \cup \beta_j < n_{i_j+1} \leq n_{i_{j+1}}$ for all $j \leq k$. Because $\mathcal{F}$ is hereditary and spreading, we infer that $\{n_{i_2}, \dots, n_{i_k}\} \in \mathcal{F}$ whence $\{n_i : i \in \mathbb{N},\, u_{n_i}(\gamma) \neq 0\} \in \mathcal{F}^+$, as required. □

Corollary 3.2. *Suppose K is homeomorphic to $[1, \omega^\xi]$, $\xi < \omega_1$, and that (f_i) is a normalized weakly null sequence in $C(K)$. Let $\mathcal{F}$ be a regular family of order ξ. Then for every $N \in [\mathbb{N}]$ and every non-increasing sequence of positive scalars (ϵ_i), there exists $M \in [N]$, $M = (m_i)$, such that for every $t \in K$ the set $\{m_i : i \in \mathbb{N},\, |f_{m_i}(t)| \geq \epsilon_i\}$ belongs to $\mathcal{F}^+$.*

Proof. We identify $C(\mathcal{F})$ with $C(K)$ and apply Proposition 3.1 to find a normalized, shrinking, monotone Schauder basis (e_i) for $C(K)$ with the following property: For every block basis $u_1 < u_2 < \dots$ of (e_i) there exist positive integers $n_1 < n_2 < \dots$ such that for all $t \in K$, $\{n_i : i \in \mathbb{N},\, u_{n_i}(t) \neq 0\} \in \mathcal{F}^+$.

Now let (f_i) be normalized weakly null in $C(K)$. A classical perturbation result [11] yields a subsequence (f_{l_i}) of (f_i) and a block basis (u_i) of (e_i), $u_1 < u_2 < \dots$, such that $l_i \in N$ and $\|f_{l_i} - u_i\| < \epsilon_i/2$, for all $i \in \mathbb{N}$. We next choose positive integers $n_1 < n_2 < \dots$ such that $\{n_i : i \in \mathbb{N},\, u_{n_i}(t) \neq 0\} \in \mathcal{F}^+$, for all $t \in K$. Set $m_i = l_{n_i}$, for all $i \in \mathbb{N}$. It is not hard to check using the spreading property of $\mathcal{F}$, that $M = (m_i)$ satisfies the desired conclusion. □

Notation 3.3. *Let $\mathcal{F}$ be a regular family and let (e_i) denote the unit vector basis of c_{00}. We define a norm $\|\cdot\|_{\mathcal{F}}$ on c_{00} by the rule*

$$\Big\|\sum_i a_i e_i\Big\|_{\mathcal{F}} = \sup\Big\{\sum_{i \in F} |a_i| : F \in \mathcal{F}\Big\}, \text{ for all } (a_i) \in c_{00}.$$

The completion of $(c_{00}, \|\cdot\|_{\mathcal{F}})$ is a Banach space having (e_i) as a normalized, unconditional, shrinking, monotone Schauder basis (see [1], [2]*). When $\mathcal{F} = S_\xi$, the ξ-th Schreier class, we obtain the generalized Schreier space X^ξ introduced in* [1], [2].

Our next result yields that every normalized weakly null sequence in $C(\omega^{\omega^\xi})$ admits a subsequence dominated by a subsequence of the unit vector basis of the generalized Schreier space X^ξ.

Proposition 3.4. *Suppose K is homeomorphic to $[1, \omega^\xi]$, $\xi < \omega_1$, and that (f_i) is a normalized weakly null sequence in $C(K)$. Let $\mathcal{F}$ be a regular family of order ξ. Given $0 < \epsilon < 1$, there exists $M \in [\mathbb{N}]$, $M = (m_i)$, such that*

$$\Big\|\sum_i a_i f_{m_i}\Big\| \leq \frac{2}{1-\epsilon} \sup\Big\{\Big\|\sum_{i \in F} a_i f_{m_i}\Big\| : F \subset \mathbb{N},\, (m_i)_{i\in F} \in \mathcal{F}\Big\}$$
$$\leq \frac{2}{1-\epsilon}\Big\|\sum_i a_i e_{m_i}\Big\|_{\mathcal{F}}, \text{ for all } (a_i) \in c_{00}.$$

Proof. We may assume that (f_i) is 2-basic. Choose a decreasing sequence of positive scalars (ϵ_i) such that $\sum_i \epsilon_i < \epsilon/3$. We next choose $M \in [\mathbb{N}]$, $M = (m_i)$, satisfying the conclusion of Corollary 3.2 applied to (f_i) and the scalar sequence (ϵ_i).

Let $(a_i) \in c_{00}$ be such that $\|\sum_i a_i f_{m_i}\| = 1$, and let $t \in K$ satisfy $|\sum_i a_i f_{m_i}(t)| = 1$. Since $\{m_i : i \in \mathbb{N}, |f_{m_i}(t)| \geq \epsilon_i\}$ belongs to $\mathcal{F}^+$, we obtain

$$1 \leq 2 \sup\Big\{\Big\|\sum_{i\in F} a_i f_{m_i}\Big\| : F \subset \mathbb{N},\, (m_i)_{i \in F} \in \mathcal{F}\Big\} + \epsilon$$

from which the assertion of the proposition follows. □

Remark 3.5. *S. Argyros has discovered an alternate proof of Corollary 3.2. He shows that given a weakly null sequence (f_i) in $C(\omega^\xi)$ and a summable sequence of positive scalars (ϵ_i) then, by identifying $C(\omega^\xi)$ with $C(\mathcal{F})$, one can select positive integers $1 = m_1 < m_2 < \ldots$ such that if $|f_{m_i}(F)| \geq \epsilon_i$ for some $i \in \mathbb{N}$ and $F \in \mathcal{F}$, then $F \cap (m_{i-1}, m_{i+1}) \neq \emptyset$ ($m_0 = 0$). Therefore, $\{m_{2i} : i \in \mathbb{N}, |f_{m_{2i}}(F)| \geq \epsilon_{2i}\} \in \mathcal{F}^+$, for every $F \in \mathcal{F}$ which clearly implies Corollary 3.2.*

Remark 3.6. *Proposition 9 and Lemma 13 in* [26] *yield that for a normalized weakly null sequence (f_i) in $C(\omega^\xi)$ there exist a subsequence (f_{m_i}), a compact hereditary family $\mathcal{D}$ with $\mathcal{D}^{(\xi+1)} = \emptyset$ and a constant $d > 0$ such that $\|\sum_i a_i f_{m_i}\| \leq d \sup\{\|\sum_{i\in A} a_i f_{m_i}\| : A \in \mathcal{D}\}$ for every $(a_i) \in c_{00}$.*

Theorem 3.7. *Suppose K is homeomorphic to $[1, \omega^\xi]$, $\xi < \omega_1$, and that (f_i) is a normalized weakly null sequence in $C(K)$. Let $\mathcal{F}$ be a regular family of order ξ. Assume (f_i) is $\mathcal{F}$-unconditional. Then (f_i) has an unconditional subsequence.*

Proof. Suppose (f_i) is $\mathcal{F}$-unconditional with constant $C > 0$. This means that $\|\sum_{i\in F} a_i f_i\| \leq C\|\sum_i a_i f_i\|$, for all $F \in \mathcal{F}$ and every $(a_i) \in c_{00}$. Let $M = (m_i)$ satisfy the conclusion of Proposition 3.4, for (f_i) and $\mathcal{F}$ with $\epsilon = 1/2$. We claim that (f_{m_i}) is unconditional. Indeed, let $(a_i) \in c_{00}$ and

$I \in [\mathbb{N}]$. Proposition 3.4 yields

$$\|\sum_{i\in I} a_i f_{m_i}\| \leq 4\sup\{\|\sum_{i\in F\cap I} a_i f_{m_i}\| : F \subset \mathbb{N},\ (m_i)_{i\in F} \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is hereditary and (f_i) is $\mathcal{F}$-unconditional, we have that

$$\|\sum_{i\in F\cap I} a_i f_{m_i}\| \leq C\|\sum_i a_i f_{m_i}\|, \text{ whenever } (m_i)_{i\in F} \in \mathcal{F}.$$

Therefore, $\|\sum_{i\in I} a_i f_{m_i}\| \leq 4C\|\sum_i a_i f_{m_i}\|$ which proves the claim. This completes the proof. $\square$

From Theorem 3.7 we easily obtain the next

Corollary 3.8. *A normalized weakly null sequence in $C(\omega^{\omega^\xi})$, $\xi < \omega_1$, admits an unconditional subsequence if, and only if, it admits a subsequence which is S_ξ-unconditional.*

Theorem 3.9. *Let (f_i) be a normalized weakly null sequence in $C(\omega^{\omega^\xi})$, $\xi < \omega_1$. Assume that (f_i) is an ℓ_1^ξ-spreading model. Then (f_i) admits a subsequence equivalent to a subsequence of the unit vector basis of X^ξ, the generalized Schreier space of order ξ (see Notation 3.3).*

We recall that $X^0 = c_0$ while X^1 was implicitly considered by Schreier [37]. The generalized Schreier spaces X^ξ, $\xi < \omega_1$, were introduced in [1], [2]. They can be thought as the the higher ordinal unconditional analogs of c_0.

We also recall ([8]), that a normalized basic sequence (x_n) is said to be an *ℓ_1^ξ-spreading model*, $\xi < \omega_1$, if there is a constant $\delta > 0$ such that $\|\sum_{n\in F} a_n x_n\| \geq \delta\sum_{n\in F}|a_n|$, for every $F \in S_\xi$ and all choices of scalars $(a_n)_{n\in F}$. Saying (x_n) is an ℓ_1^1-spreading model means that ℓ_1 is a spreading model for the space generated by some subsequence of (x_n), in the sense of [14], [9], [31]. ℓ_1^ξ-spreading models are instrumental in the study of *asymptotic* ℓ_1-spaces [33]. It is shown in [6] that a weakly null sequence which is an ℓ_1^ξ-spreading model, admits a subsequence which is S_ξ-unconditional. The unit vector basis of X^ξ is an ℓ_1^ξ-spreading model with constant $\delta = 1$.

Proof of Theorem 3.9. We first apply Proposition 3.4 with $\epsilon = 1/2$, to obtain an infinite subset $M = (m_i)$ of $\mathbb{N}$ with $\|\sum_i a_i f_{m_i}\| \leq 4\|\sum_i a_i e_{m_i}\|_{S_\xi}$ for all $(a_i) \in c_{00}$, where (e_i) denotes the unit vector basis of X^ξ. On the other hand, as (f_i) is an ℓ_1^ξ-spreading model, there exists a constant $\delta > 0$ such that

$$\|\sum_i a_i f_{m_i}\| \geq \delta\|\sum_i a_i e_{m_i}\|_\xi, \text{ for all } (a_i) \in c_{00}.$$

We infer from the preceding inequalities that (f_{m_i}) and (e_{m_i}) are equivalent. $\square$

Our final result in this section yields a quantitative version of Rosenthal's result, that a weakly null (in $C(K)$) sequence of indicator functions

of clopen subsets of a compact Hausdorff space K, admits an unconditional subsequence (cf. also [8] and [7] for another proof of this result).

Theorem 3.10. *Let K be a compact Hausdorff space. Suppose that (f_n) is a normalized weakly null sequence in $C(K)$ such that there exists $\epsilon > 0$ with the property $f_n(t) = 0$ or $|f_n(t)| \geq \epsilon$ for all $t \in K$ and $n \in \mathbb{N}$. Then there exist $\xi < \omega_1$ and a subsequence of (f_n) equivalent to a subsequence of the natural Schauder basis of X^ξ.*

Proof. We first employ the results of [1] in order to find the smallest countable ordinal η for which there is a subsequence (f_{m_n}) of (f_n), such that no subsequence of (f_{m_n}) is an ℓ_1^η-spreading model. Such an ordinal exists because (f_n) is weakly null. We claim that η is a successor ordinal. To see this we shall need a result from [6] (Corollary 3.6) which states that a weakly null sequence (f_n) in a $C(K)$ space admits a subsequence which is an ℓ_1^α-spreading model, for some $\alpha < \omega_1$ if, and only if, there exist a constant $\delta > 0$ and $L \in [\mathbb{N}]$, $L = (l_n)$, so that for every $F \in S_\alpha$ there exists $t \in K$ satisfying $|f_{l_n}(t)| \geq \delta$, for all $n \in F$.

Define $G_n = \{t \in K : f_n(t) \neq 0\}$. Our assumptions yield $G_n = \{t \in K : |f_n(t)| \geq \epsilon\}$, for all $n \in \mathbb{N}$. Observe that for every $\alpha < \eta$ and $P \in [\mathbb{N}]$, there exists $Q \in [P]$, $Q = (q_n)$, so that (f_{q_n}) is an ℓ_1^α-spreading model. It follows now, from the previously cited result of [6], that for every $\alpha < \eta$ and $P \in [\mathbb{N}]$, there exists $Q \in [P]$, $Q = (q_n)$, so that for every $F \in S_\alpha$, $\cap_{n\in F} G_{q_n} \neq \emptyset$. This in turn yields that every subsequence of (f_{m_n}) admits, for every $\alpha < \eta$, a further subsequence which is an ℓ_1^α-spreading model with constant independent of α and the particular subsequence. Were η a limit ordinal, we would have that some subsequence of (f_{m_n}) is an ℓ_1^η-spreading model, contrary to our assumption.

Hence, $\eta = \xi + 1$, for some $\xi < \omega_1$. Let (e_n) be the natural basis of X^ξ. We show that some subsequence of (f_{m_n}) is equivalent to a subsequence of (e_n). Because $\xi < \eta$, we can assume without loss of generality, after passing to a subsequence if necessary, that (f_{m_n}) is an ℓ_1^ξ-spreading model and thus there exists a constant $\rho > 0$ such that $\|\sum_n a_n f_{m_n}\| \geq \rho \|\sum_n a_n e_n\|_{S_\xi}$ for all $(a_n) \in c_{00}$. Define

$$\mathcal{F} = \{F \in [\mathbb{N}]^{<\infty} : \cap_{i\in F} G_{m_i} \neq \emptyset\}.$$

Clearly, $\mathcal{F}$ is hereditary. It is shown in [6], based on the fact that no subsequence of (f_{m_n}) is an $\ell_1^{\xi+1}$-spreading model, that there exist $L \in [\mathbb{N}]$, $L = (l_n)$, and $d \in \mathbb{N}$ so that every member of $\mathcal{F}[L]$ is contained in the union of d members of $S_\xi[L]$. Let $k_n = m_{l_n}$, for all $n \in \mathbb{N}$. We deduce from our preceding work that $\|\sum_n a_n f_{k_n}\| \leq d \|\sum_n a_n e_{l_n}\|_{S_\xi}$, for every $(a_n) \in c_{00}$. Therefore, (f_{k_n}) and (e_{l_n}) are equivalent. $\square$

4. Normalized averages of a basic sequence

Let $\vec{s} = (e_n)$ be a normalized basic sequence in a Banach space, and let $\mathcal{F}$ be a regular and stable family. We shall introduce an hierarchy

$\{(\alpha_n^{\mathcal{F},\vec{s},M})_{n=1}^\infty,\ M \in [\mathbb{N}],\ \alpha < \omega_1\}$ of normalized block bases of $\vec{s}$, similar to that of the repeated averages introduced in [8]. The latter however consists of convex block bases of $\vec{s}$, not necessarily normalized.

We fix a normalized basic sequence $\vec{s} = (e_n)$ and a regular and stable family $\mathcal{F}$. To simplify our notation, we shall write α_n^M instead of $\alpha_n^{\mathcal{F},\vec{s},M}$. We shall next define, by transfinite induction on $\alpha < \omega_1$, a family of normalized block bases $(\alpha_n^M)_{n=1}^\infty$ of $\vec{s}$, where $M \in [\mathbb{N}]$, so that the following properties are fulfilled for every $\alpha < \omega_1$ and $M \in [\mathbb{N}]$:

(1) $\alpha_n^M < \alpha_{n+1}^M$, for all $n \in \mathbb{N}$.
(2) $M = \cup_n \mathrm{supp}\, \alpha_n^M$, for all $M \in [\mathbb{N}]$.

If $\alpha = 0$ and $M = (m_n)$ set $\alpha_n^M = e_{m_n}$, for all $n \in \mathbb{N}$.

Suppose $(\beta_n^N)_{n=1}^\infty$ has been defined so that (1) and (2), above, are satisfied for all $\beta < \alpha$ and $N \in [\mathbb{N}]$. Let $M \in [\mathbb{N}]$. In order to define $(\alpha_n^M)_{n=1}^\infty$, assume first that α is successor, say $\alpha = \beta + 1$. Let k_1 be the unique integer such that the set $\{\min \mathrm{supp}\, \beta_i^M : i \le k_1\}$ is a maximal member of $\mathcal{F}$. We define

$$\alpha_1^M = \Big(\sum_{i=1}^{k_1} \beta_i^M\Big) \,/\, \Big\|\sum_{i=1}^{k_1} \beta_i^M\Big\|.$$

Suppose that $\alpha_1^M < \cdots < \alpha_n^M$ have been defined and that the union of their supports forms an initial segment of M. Set

$$M_{n+1} = \{m \in M :\ \max \mathrm{supp}\, \alpha_n^M < m\}.$$

Let k_{n+1} be the unique integer such that the set $\{\min \mathrm{supp}\, \beta_i^{M_{n+1}} : i \le k_{n+1}\}$ is a maximal member of $\mathcal{F}$. We define

$$\alpha_{n+1}^M = \Big(\sum_{i=1}^{k_{n+1}} \beta_i^{M_{n+1}}\Big) \,/\, \Big\|\sum_{i=1}^{k_{n+1}} \beta_i^{M_{n+1}}\Big\|.$$

This completes the definition of $(\alpha_n^M)_{n=1}^\infty$ when α is a successor ordinal. Note that the construction described above can be carried out because $\mathcal{F}$ is stable. (1) and (2) are now satisfied by $(\alpha_n^M)_{n=1}^\infty$.

Now suppose α is a limit ordinal. Let $(\alpha_n + 1)$ be the sequence of successor ordinals associated to α. Let $M \in [\mathbb{N}]$ and set $m_1 = \min M$. In case $m_1 = 1$, set $\alpha_1^M = e_1$. If $m_1 > 1$, define

$$\alpha_1^M = u^M \,/\, \|u^M\|, \text{ where } u^M = (1/m_1)e_{m_1} + [\alpha_{m_1}]_1^{M \setminus \{m_1\}}.$$

Suppose that $\alpha_1^M < \cdots < \alpha_n^M$ have been defined and that the union of their supports forms an initial segment of M. Set

$$M_{n+1} = \{m \in M :\ \max \mathrm{supp}\, \alpha_n^M < m\}$$

and $m_{n+1} = \min M_{n+1}$. Define

$$\alpha_{n+1}^M = u^{M_{n+1}} \,/\, \|u^{M_{n+1}}\|, \text{ where}$$
$$u^{M_{n+1}} = (1/m_{n+1})e_{m_{n+1}} + [\alpha_{m_{n+1}}]_1^{M_{n+1} \setminus \{m_{n+1}\}}.$$

Note that $\alpha^M_{n+1} = \alpha_1^{M_{n+1}}$. This completes the definition $(\alpha^M_n)_{n=1}^\infty$ when α is a limit ordinal. It is clear that (1) and (2) are satisfied.

Remark 4.1. *In case $\mathcal{F} = S_1$, the first Schreier family, it is not hard to see that* $\operatorname{supp} \alpha^M_n \in S_\alpha$*, for all $\alpha < \omega_1$, all $M \in [\mathbb{N}]$ and all $n \in \mathbb{N}$.*

The next lemma is an immediate consequence of the preceding definition.

Lemma 4.2. *Let $\alpha < \omega_1$, $M \in [\mathbb{N}]$ and $n \in \mathbb{N}$. Then there exists $N \in [\mathbb{N}]$ such that $\alpha^M_n = \alpha^N_1$.*

Our next result will be applied later, in conjunction with the infinite Ramsey theorem, in order to determine if there exists a block basis of the form (α^M_n), equivalent to the c_0-basis.

Lemma 4.3. *Let $\alpha < \omega_1$, $M \in [\mathbb{N}]$ and $n \in \mathbb{N}$. Let $L_i \in [\mathbb{N}]$ and $k_i \in \mathbb{N}$, for $i \leq n$, be so that $\alpha^{L_1}_{k_1} < \cdots < \alpha^{L_n}_{k_n}$ and $\cup_{i=1}^n \operatorname{supp} \alpha^{L_i}_{k_i}$ is an initial segment of M. Then $\alpha^M_i = \alpha^{L_i}_{k_i}$, for all $i \leq n$.*

Proof. By Lemma 4.2 we may assume that $k_i = 1$ for all $i \leq n$. We prove the assertion of the lemma by transfinite induction on α. The case $\alpha = 0$ is trivial. Suppose the assertion holds for all ordinals smaller than α, and all $M \in [\mathbb{N}]$ and $n \in \mathbb{N}$. Let $M \in [\mathbb{N}]$. We prove the assertion for α by induction on n. If $n = 1$, we first consider the case of α being a successor ordinal, say $\alpha = \beta + 1$. We know from the definitions that

$$\operatorname{supp} \alpha^{L_1}_1 = \cup_{j=1}^{p_1} \operatorname{supp} \beta^{L_1}_j,$$

where $\{\min \operatorname{supp} \beta^{L_1}_j : j \leq p_1\}$ is a maximal member of $\mathcal{F}$. In particular, the set $\cup_{j=1}^{p_1} \operatorname{supp} \beta^{L_1}_j$ is an initial segment of M. The induction hypothesis on β now implies that $\beta^M_j = \beta^{L_1}_j$, for all $j \leq p_1$. It follows now that $\alpha^M_1 = \alpha^{L_1}_1$.

To complete the case $n = 1$, we consider the possibility that α is a limit ordinal. Let $(\alpha_n + 1)$ be the sequence of ordinals associated to α and suppose that $m = \min M$. Then $m = \min \operatorname{supp} \alpha^{L_1}_1$ and so $m = \min L_1$. In case $m = 1$ we have, trivially, $\alpha^M_1 = \alpha^{L_1}_1 = e_m$. When $m > 1$, $u^M = (1/m)e_m + [\alpha_m]_1^{M \setminus \{m\}}$, $u^{L_1} = (1/m)e_m + [\alpha_m]_1^{L_1 \setminus \{m\}}$ and $\alpha^M_1 = u^M / \|u^M\|$, $\alpha^{L_1}_1 = u^{L_1} / \|u^{L_1}\|$.

It follows that $\operatorname{supp} [\alpha_m]_1^{L_1 \setminus \{m\}}$ is an initial segment of $M \setminus \{m\}$, and so we infer from the induction hypothesis applied to α_m, that $[\alpha_m]_1^{L_1 \setminus \{m\}} = [\alpha_m]_1^{M \setminus \{m\}}$. Thus $\alpha^M_1 = \alpha^{L_1}_1$ which completes the case $n = 1$.

Assume now the assertion holds for $n-1$ and write $M = \cup_{i=1}^n \operatorname{supp} \alpha^{L_i}_1 \cup N$, where $\cup_{i=1}^n \operatorname{supp} \alpha^{L_i}_1$ is an initial segment of M, which is disjoint from N. The induction hypothesis for $n-1$ yields $\alpha^M_i = \alpha^{L_i}_1$ for all $i < n$. Hence $M = \cup_{i=1}^{n-1} \operatorname{supp} \alpha^M_i \cup P$, where $P = \operatorname{supp} \alpha^{L_n}_1 \cup N$. It follows from the definition that $\alpha^M_n = \alpha^P_1$. But now the case $n = 1$ guarantees that $\alpha^P_1 = \alpha^{L_n}_1$ and the assertion of the lemma is settled. $\square$

Terminology. Let (e_n) be a normalized Schauder basic sequence in a Banach space and let $\mathcal{F}$ be a regular family. A finite block basis $u_1 < \cdots < u_m$ of (e_n) is said to be $\mathcal{F}$-*admissible* if $\{\min \operatorname{supp} u_i : i \leq m\} \in \mathcal{F}$. It is called *maximally* $\mathcal{F}$-admissible, if $\mathcal{F}$ is additionally assumed to be stable and $\{\min \operatorname{supp} u_i : i \leq m\}$ is a maximal member of $\mathcal{F}$.

Definition 4.4. *A normalized block basis (u_n) of (e_n) with $u_1 < u_2 < \dots$ is a c_0^{ξ}-spreading model, if there exists a constant $C > 0$ such that $\| \sum_{i \in F} a_i u_i \| \leq C \max_{i \in F} |a_i|$, for every $F \in [\mathbb{N}]^{<\infty}$ with $(u_i)_{i \in F}$ S_{ξ}-admissible, and every choice of scalars $(a_i)_{i \in F}$.*

In what follows we fix a normalized basic sequence $\vec{s} = (e_n)$ and a regular and stable family $\mathcal{F}$. We abbreviate $\alpha_n^{\mathcal{F}, \vec{s}, M}$ to α_n^M.

Terminology. Suppose that $\alpha < \omega_1$ and $M \in [\mathbb{N}]$. An α-average of (e_n) supported by M, is any vector of the form α_1^L for some $L \in [M]$.

In the sequel we shall make use of the infinite Ramsey theorem [17], [31] and so we recall its statement. $[\mathbb{N}]$ is endowed with the topology of pointwise convergence.

Theorem 4.5. *Let $\mathcal{A}$ be an analytic subset of $[\mathbb{N}]$. Then there exists $N \in [\mathbb{N}]$ so that either $[N] \subset \mathcal{A}$, or $[N] \cap \mathcal{A} = \emptyset$.*

Our next result is inspired by an unpublished result of W.B. Johnson (see [31]).

Lemma 4.6. *Let α and γ be countable ordinals and suppose there exists $N \in [\mathbb{N}]$ such that for every $M \in [N]$ there exists a block basis of α-averages of (e_n), supported by M, which is a c_0^{γ}-spreading model. Then there exist $M \in [N]$ and a constant $C > 0$ so that $\| \sum_{i=1}^{n_L} \alpha_i^L \| \leq C$, for every $L \in [M]$, where n_L stands for the unique integer satisfying $\{\min \operatorname{supp} \alpha_i^L : i \leq n_L\}$ is maximal in S_{γ}.*

Proof. Define $\mathcal{D}_k = \{L \in [N] : \| \sum_{i=1}^{n_L} \alpha_i^L \| \leq k\}$, for all $k \in \mathbb{N}$. $\mathcal{D}_k$ is closed in the topology of pointwise convergence, thanks to Lemma 4.3. We claim that there exist $k \in \mathbb{N}$ and $M \in [N]$ so that $[M] \subset \mathcal{D}_k$. The assertion of the lemma clearly follows once this claim is established. Were the claim false, then Theorem 4.5 would yield a nested sequence $M_1 \supset M_2 \supset \dots$ of infinite subsets of N such that $[M_k] \cap \mathcal{D}_k = \emptyset$, for all $k \in \mathbb{N}$. Choose an infinite sequence of integers $m_1 < m_2 < \dots$ with $m_i \in M_i$ for all $i \in \mathbb{N}$. Set $M = (m_i)$. Since $M \in [N]$ our assumptions yield a block basis (u_i) of α-averages of (e_i), supported by M, which is a c_0^{γ}-spreading model. Therefore there exists a constant $C > 0$ such that $\| \sum_{i \in F} u_i \| \leq C$, whenever $(u_i)_{i \in F}$ is S_{γ}-admissible. Choose $k \in \mathbb{N}$ with $k > C$. Then choose $i_0 \in \mathbb{N}$ so that $\operatorname{supp} u_i \subset M_k$, for all $i > i_0$. If we set $L = \cup_{i=i_0+1}^{\infty} \operatorname{supp} u_i$, then $L \in [M_k]$, and $\alpha_i^L = u_{i+i_0}$, for all $i \in \mathbb{N}$, by Lemma 4.3. Hence, $L \notin \mathcal{D}_k$. However,

$$\| \sum_{i=1}^{n_L} \alpha_i^L \| = \| \sum_{i=1}^{n_L} u_{i_0+i} \| \leq C < k$$

which is a contradiction. □

5. Convolution of transfinite averages

We fix a normalized 2-basic, shrinking sequence $\vec{s} = (e_i)$ in some Banach space. We shall often make use of the following result established in [32]: Given $\epsilon > 0$ there exists $M \in [\mathbb{N}]$ such that for every finitely supported scalar sequence $(a_i)_{i\in M}$ with $\|\sum_{i\in M} a_i e_i\| = 1$, we have $\max_{i\in M} |a_i| \leq 1 + \epsilon$. For the rest of this section, we let $\mathcal{F} = S_1$. *All transfinite averages of $\vec{s}$ will be taken with respect to $\mathcal{F}$.* As in the previous section, α_n^M abbreviates $\alpha_n^{\mathcal{F},\vec{s},M}$.

The purpose of the present section is to deal with the following problem: Let α and β be countable ordinals and suppose that (u_i) is a block basis of $(\alpha+\beta)$-averages of $\vec{s}$. Does there exist a block basis (v_i) of α-averages of $\vec{s}$ such that (u_i) is a block basis of β-averages of (v_i) ?

It follows directly from the definitions that this is indeed the case when $\beta < \omega$. However, if β is an infinite ordinal, the preceding question has, in general, a negative answer.

In Proposition 5.9, we give a partially affirmative answer to this question which, roughly speaking, states that every $(\alpha+\beta)$ average of $\vec{s}$ can be represented as a finite sum $\sum_{i=1}^n \lambda_i w_i$, where $w_1 < \cdots < w_n$ is an S_β-admissible block basis of α-averages of $\vec{s}$ and $(\lambda_i)_{i=1}^n$ is a sequence of positive scalars which are almost equal each other. We employ this result in order to prove the following theorem about transfinite c_0-spreading models of $\vec{s}$, which will in turn be applied in subsequent sections. In the sequel, when we refer to a block basis we shall always mean a block basis of $\vec{s}$. Also all transfinite averages will be taken with respect to $\vec{s}$.

Theorem 5.1. *Let α and β be countable ordinals and $N \in [\mathbb{N}]$. Suppose that for every $P \in [N]$ there exists $M \in [P]$ such that no block basis of α-averages supported by M is a c_0^β-spreading model. Then for every $P \in [N]$ and $\epsilon > 0$ there exists $Q \in [P]$ with the following property: Every $(\alpha+\beta)$-average u supported by Q admits a decomposition $u = \sum_{i=1}^n \lambda_i u_i$, where $u_1 < \cdots < u_n$ is a normalized block basis and $(\lambda_i)_{i=1}^n$ is a sequence of positive scalars such that*

1. *There exists $I \subset \{1,\dots,n\}$ with $(u_i)_{i\in I}$ S_β-admissible, and such that u_i is an α-average for all $i \in I$, while $\|\sum_{i\in\{1,\dots,n\}\setminus I} \lambda_i u_i\|_{\ell_1} < \epsilon$.*
2. *$\max_{i\in I} \lambda_i < \epsilon$.*

Recall that if $\sum_{i=1}^n a_i e_i$ is a finite linear combination of $\vec{s}$ then we denote by $\|\sum_{i=1}^n a_i e_i\|_{\ell_1}$ the quantity $\sum_{i=1}^n |a_i|$. To prove this theorem we shall need to introduce some terminology.

Definition 5.2. *Let α and β be countable ordinals and $\epsilon > 0$. A normalized block u is said to admit an (ϵ,α,β)-decomposition, if there exist normalized blocks $u_1 < \cdots < u_n$ and positive scalars $(\lambda_i)_{i=1}^n$ with $u = \sum_{i=1}^n \lambda_i u_i$ and so that the following conditions are satisfied:*

(1) *There exists* $I \subset \{1, \dots, n\}$ *with* $(u_i)_{i \in I}$ S_β*-admissible, and such that* u_i *is an* α*-average for all* $i \in I$*, while* $\|\sum_{i \in \{1,\dots,n\} \setminus I} \lambda_i u_i\|_{\ell_1} < \epsilon$*.*
(2) $|\lambda_i - \lambda_j| < \epsilon$ *for all* i *and* j *in* I*.*

Terminology. The quantity $\max_{i \in I} \lambda_i$ is called the *weight* of the decomposition. If u is an $(\alpha + \beta)$-average admitting an $(\epsilon, \alpha, \beta)$-decomposition, $u = \sum_{i=1}^n \lambda_i u_i$, satisfying (1), (2), above, and $I \subset \{1, \dots, n\}$ is as in (1), then every subset of $\{\min \operatorname{supp} u_i : i \in I\}$ will be called an $(\epsilon, \alpha, \beta)$-*admissible subset of* $\mathbb{N}$ *resulting from* u. It is clear that the collection of all $(\epsilon, \alpha, \beta)$-admissible subsets of $\mathbb{N}$ resulting from some (not necessarily the same) $(\alpha + \beta)$-average (for some fixed choices of ϵ, α, β), forms a hereditary family.

Lemma 5.3. *Let* $P \in [\mathbb{N}]$ *and* $\epsilon > 0$*. Assume that for every* $L \in [P]$ *there exists an* $(\alpha + \beta)$*-average supported by* L *which admits an* $(\epsilon, \alpha, \beta)$*-decomposition. Then there exists* $Q \in [P]$ *such that every* $(\alpha + \beta)$*-average supported by* Q *admits an* $(\epsilon, \alpha, \beta)$*-decomposition.*

Proof. Let

$$\mathcal{D} = \{L \in [P] : [\alpha + \beta]_1^L \text{ admits an } (\epsilon, \alpha, \beta) - \text{decomposition}\}.$$

Lemma 4.3 yields that $\mathcal{D}$ is closed in the topology of pointwise convergence. Theorem 4.5 now implies the existence of some $Q \in [P]$ such that either $[Q] \subset \mathcal{D}$, or $[Q] \cap \mathcal{D} = \emptyset$. Our assumptions rule out the second alternative for Q. Hence $[Q] \subset \mathcal{D}$ which proves the lemma. □

In the next series of lemmas (Lemma 5.4 and Lemma 5.5), we describe some criteria for embedding a Schreier family into an appropriate hereditary family of finite subsets of $\mathbb{N}$. These criteria, as well as their proofs, are variants of similar results contained in [8], [6]. We shall therefore omit the proofs and refer the reader to the aforementioned papers (see for instance Propositions 2.3.2 and 2.3.6 in [8], or Theorems 2.11 and 2.13 in [6]). These lemmas will be applied in the proof of Proposition 5.9, which constitutes the main step towards the proof of Theorem 5.1.

Notation. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ and $M \in [\mathbb{N}]$. Let $M = (m_i)$ be the increasing enumeration of M. We set $\mathcal{F}(M) = \big\{\{m_i : i \in F\} : F \in \mathcal{F}\big\}$. Clearly, $\mathcal{F}(M) \subset \mathcal{F}$ if $\mathcal{F}$ is spreading. We also recall that $\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\infty}$. Finally, for every $L \in [\mathbb{N}]$ and $\alpha < \omega_1$, we let $(F_i^\alpha(L))_{i=1}^\infty$ denote the unique decomposition of L into successive, maximal members of S_α.

Lemma 5.4. *Suppose that* $1 \leq \xi < \omega_1$*,* $\mathcal{D}$ *is a hereditary family of finite subsets of* $\mathbb{N}$ *and* $N \in [\mathbb{N}]$*. Assume that for every* $n \in \mathbb{N}$ *and* $P \in [N]$ *there exists* $L \in [P]$ *such that* $\cup_{i=1}^n (F_i^\xi(L) \setminus \{\min F_i^\xi(L)\}) \in \mathcal{D}$*. Then there exists* $M \in [N]$ *such that* $S_{\xi+1}(M) \subset \mathcal{D}$*.*

Lemma 5.5. *Suppose that* $\mathcal{D}$ *is a hereditary family of finite subsets of* $\mathbb{N}$ *and* $N \in [\mathbb{N}]$*. Let* $\xi < \omega_1$ *be a limit ordinal and let* (α_n) *be an increasing sequence*

of ordinals tending to ξ. Assume there exists a sequence $M_1 \supset M_2 \supset \dots$ of infinite subsets of N such that $S_{\alpha_n}(M_n) \subset \mathcal{D}$, for all $n \in \mathbb{N}$. Then there exists $M \in [N]$ such that $S_\xi(M) \subset \mathcal{D}$.

In the sequel we shall make use of the following permanence property of Schreier families established in [33]:

Lemma 5.6. *Suppose that $\alpha < \beta < \omega_1$. Then there exists $n \in \mathbb{N}$ such that for every $F \in S_\alpha$ with $n \leq \min F$ we have $F \in S_\beta$.*

We shall also make repeated use of the following result from [5]:

Lemma 5.7. *For every $N \in [\mathbb{N}]$ there exists $M \in [N]$ such that for every $\alpha < \omega_1$ and $F \in S_\alpha[M]$ we have $F \setminus \{\min F\} \in S_\alpha(N)$.*

Lemma 5.7 combined with Proposition 3.2 in [33] yields the next

Lemma 5.8. *Let α and β be countable ordinals and $N \in [\mathbb{N}]$. Then there exists $M \in [N]$ such that*

(1) *For every $F \in S_\beta[S_\alpha][M]$ we have $F \setminus \{\min F\} \in S_{\alpha+\beta}$.*
(2) *For every $F \in S_{\alpha+\beta}[M]$ we have $F \setminus \{\min F\} \in S_\beta[S_\alpha]$.*

Proposition 5.9. *Let α and β be countable ordinals and $N \in [\mathbb{N}]$. Then given $\epsilon > 0$ and $P \in [N]$ there exist $Q \in [P]$ and $R \in [Q]$ such that*

(1) *Every $(\alpha + \beta)$-average supported by Q admits an $(\epsilon, \alpha, \beta)$ decomposition.*
(2) *For every $F \in S_\beta[R]$, $F \setminus \{\min F\}$ is an $(\epsilon, \alpha, \beta)$-admissible set resulting from some $(\alpha + \beta)$-average supported by Q.*

Proof. Fix $\alpha < \omega_1$. We prove the assertion of the proposition by transfinite induction on β. The case $\beta = 1$ follows directly from the definitions since every $(\alpha+1)$-average admits an $(\epsilon, \alpha, 1)$-decomposition. In fact, in this case, we may take $Q = P$ and $R = \{\min \operatorname{supp} \alpha_i^P : i \in \mathbb{N}\}$ and check that (1) and (2) hold.

Now let $\beta > 1$ and suppose the assertion holds for all ordinals smaller than β. Assume first β is a successor ordinal and let $\beta - 1$ be its predecessor. Let $\epsilon > 0$ and $P \in [N]$ be given and choose a sequence of positive scalars (δ_i) such that $\sum_i \delta_i < \epsilon/4$. Let $M \in [P]$. The induction hypothesis for $\beta - 1$ yields infinite subsets $R_1 \subset Q_1$ of M satisfying (1) and (2) for $(\delta_1, \alpha, \beta - 1)$. Choose a maximal member F_1 of $S_{\beta-1}$ with $F_1 \subset R_1$. We may choose an $(\alpha + \beta - 1)$-average u_1, supported by Q_1 and such that $F_1 \setminus \{\min F_1\}$ is $(\delta_1, \alpha, \beta - 1)$-admissible resulting from u_1.

Choose $M_2 \in [M]$ with $\min M_2 > \max \operatorname{supp} u_1$. Arguing similarly, we choose a maximal member F_2 of $S_{\beta-1}$ with $F_2 \subset M_2$, and an $(\alpha + \beta - 1)$-average u_2 supported by M_2, which admits a $(\delta_2, \alpha, \beta - 1)$-decomposition from which $F_2 \setminus \{\min F_2\}$ is resulting. We continue in this fashion and obtain a sequence $F_1 < F_2 < \dots$, of successive maximal members of $S_{\beta-1}[M]$, and a block basis $u_1 < u_2 < \dots$, of $(\alpha + \beta - 1)$-averages supported by M such

that for all $i \in \mathbb{N}$,

$$u_i \text{ admits a } (\delta_i, \alpha, \beta - 1) - \text{decomposition.} \tag{5.1}$$

$$F_i \setminus \{\min F_i\} \text{ is } (\delta_i, \alpha, \beta - 1) - \text{admissible, resulting from } u_i. \tag{5.2}$$

We next let, for all $i \in \mathbb{N}$, d_i denote the weight of the $(\delta_i, \alpha, \beta - 1)$-decomposition of u_i, from which $F_i \setminus \{\min F_i\}$ is resulting. Clearly, $d_i \in (0, 3]$. Therefore, without loss of generality, by passing to a subsequence if necessary, we may assume that

$$|d_i - d_j| < \epsilon/4, \text{ for all } i, j \text{ in } \mathbb{N}. \tag{5.3}$$

Now let $n \in \mathbb{N}$ and choose $n < i_1 < \cdots < i_m$ such that $(u_{i_k})_{k=1}^m$ is maximally S_1-admissible. Set $u = (\sum_{k=1}^m u_{i_k}) \,/\, \| \sum_{k=1}^m u_{i_k} \|$. It is clear that u is an $(\alpha + \beta)$-average supported by M. It is easy to check, using (5.1) and (5.3), that u admits an $(\epsilon, \alpha, \beta)$-decomposition. On the other hand, (5.2) implies that $\cup_{k=1}^n (F_{i_k} \setminus \{\min F_{i_k}\})$ is $(\epsilon, \alpha, \beta)$-admissible, resulting from u.

Taking in account the stability of $S_{\beta-1}$, we conclude the following: Given $n \in \mathbb{N}$ and $M \in [P]$

$$\begin{aligned} &\text{There exists an } (\alpha + \beta) - \text{average } u \text{ supported by } M \\ &\text{which admits an } (\epsilon, \alpha, \beta) - \text{decomposition.} \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\text{There exists } L \in [M] \text{ such that } \cup_{i=1}^n \,(F_i^{\beta-1}(L) \setminus \{\min F_i^{\beta-1}(L)\}) \\ &\text{is } (\epsilon, \alpha, \beta) - \text{admissible, resulting from } u. \end{aligned} \tag{5.5}$$

(Recall that for $\gamma < \omega_1$, $(F_i^\gamma(L))_{i=1}^\infty$ denotes the unique decomposition of L into consecutive, maximal members of S_γ).

Lemma 5.3 and (5.4) now yield some $Q \in [P]$ satisfying (1) for $(\epsilon, \alpha, \beta)$. Let $\mathcal{D}$ denote the hereditary family of the $(\epsilon, \alpha, \beta)$-admissible subsets of Q resulting from some $(\alpha + \beta)$-average supported by Q. We infer from (5.5) that for every $n \in \mathbb{N}$ and $M \in [Q]$ there exists $L \in [M]$ such that $\cup_{i=1}^n (F_i^{\beta-1}(L) \setminus \{\min F_i^{\beta-1}(L)\}) \in \mathcal{D}$. We deduce from Lemma 5.4 that $S_\beta(R_0) \subset \mathcal{D}$ for some $R_0 \in [Q]$. Employing Lemma 5.7, we find $R \in [R_0]$ such that $F \setminus \{\min F\} \in \mathcal{D}$, for all $F \in S_\beta[R]$. Thus Q and R satisfy (1) and (2) for $(\epsilon, \alpha, \beta)$, when β is a successor ordinal.

We now consider the case of β being a limit ordinal. We may choose an increasing sequence of ordinals (β_n) having β as its limit, and such that $(\alpha + \beta_n + 1)$ is the sequence of successor ordinals associated to the limit ordinal $\alpha + \beta$. Let $\epsilon > 0$ and $P \in [N]$ be given. Let $M \in [P]$ and choose $m \in M$ with $1/m < \epsilon/4$. Next choose $M_1 \in [M]$ with $m < \min M_1$ and such that $S_{\beta_m}[M_1] \subset S_\beta$ (see Lemma 5.6). We now apply the induction hypothesis for β_m to obtain an $(\alpha + \beta_m)$-average v supported by M_1 and admitting an $(\epsilon/4, \alpha, \beta_m)$-decomposition. It is clear that $u = ((1/m)e_m + v) \,/\, \|(1/m)e_m + v\|$, is an $(\alpha + \beta)$-average supported by M and admitting

an (ϵ,α,β)-decomposition. Note also that if F is $(\epsilon/4,\alpha,\beta_m)$-admissible resulting from v, then it is also (ϵ,α,β)-admissible resulting from u.

It follows now, by lemma 5.3, that there exists $Q \in [P]$ such that (1) holds for (ϵ,α,β). Next choose positive integers $k_1 < k_2 < \dots$ such that $S_{\beta_n}[k_n,\infty) \subset S_\beta$ (see Lemma 5.6), for all $n \in \mathbb{N}$. Successive applications of the inductive hypothesis applied to each β_n and Lemma 5.7, yield infinite subsets $Q_1 \supset R_1 \supset Q_2 \supset R_2 \supset \dots$ of Q with $k_n < \min Q_n$ and such that each member of $S_{\beta_n}(R_n)$ is an $(\epsilon/4,\alpha,\beta_n)$-admissible set resulting from some $(\alpha+\beta_n)$-average supported by Q_n, for all $n \in \mathbb{N}$. Let $\mathcal{D}$ denote the hereditary family of the (ϵ,α,β)-admissible subsets of Q resulting from some $(\alpha+\beta)$-average supported by Q. Our preceding argument shows that $S_{\beta_n}(R_n) \subset \mathcal{D}$, as long as $n \in Q$ and $1/n < \epsilon/4$. We deduce now from Lemma 5.5, that there exists $R_0 \in [Q]$ such that $S_\beta(R_0) \subset \mathcal{D}$. Once again, Lemma 5.7 yields some $R \in [R_0]$ with the property $F \setminus \{\min F\} \in \mathcal{D}$, for all $F \in S_\beta[R]$. Hence, $Q \supset R$ satisfy (1) and (2) for (ϵ,α,β), when β is a limit ordinal. This completes the inductive step and the proof of the proposition. $\square$

In the proof of Theorem 5.1 we shall need Elton's nearly unconditional theorem ([18], [31]).

Theorem 5.10. *Let (f_i) be a normalized weakly null sequence in some Banach space. There exists a subsequence (f_{m_i}) of (f_i) with the following property: For every $0 < \delta \le 1$ there exists a constant $C(\delta) > 0$ such that $\|\sum_{i\in F} a_i f_{m_i}\| \le C(\delta)\|\sum_i a_i f_{m_i}\|$, for every finitely supported scalar sequence (a_i) in $[-1,1]$ and every $F \subset \{i \in \mathbb{N} : |a_i| \ge \delta\}$.*

Proof of Theorem 5.1. Let $P \in [N]$ and $\epsilon > 0$. Set

$$\begin{aligned}\mathcal{D} = &\{L \in [P] : [\alpha+\beta]_1^L \text{ admits an} \\ &\quad (\epsilon,\alpha,\beta) - \text{decomposition of weight smaller than } \epsilon\}.\end{aligned}$$

Lemma 4.3 yields that $\mathcal{D}$ is closed in the topology of pointwise convergence. The theorem asserts that $[Q] \subset \mathcal{D}$, for some $Q \in [P]$. Suppose this is not the case and choose, according to Theorem 4.5, $Q_0 \in [P]$ such that $[Q_0] \cap \mathcal{D} = \emptyset$. Next choose $Q_1 \in [Q_0]$ such that no block basis of α-averages supported by Q_1 is a c_0^β-spreading model. Let $M \in [Q_1]$. We infer from Proposition 5.9 that there exist infinite subsets $R \subset Q$ of M such that

$$\text{Every } (\alpha+\beta) - \text{average supported by } Q \text{ admits an } (\epsilon/2,\alpha,\beta) - \text{decomposition.} \tag{5.6}$$

$$\text{If } F \in S_\beta[R], \text{ then } F \setminus \{\min F\} \text{ is } (\epsilon/2,\alpha,\beta) - \text{admissible resulting from some } (\alpha+\beta) - \text{average supported by } Q. \tag{5.7}$$

Choose a maximal member F of $S_\beta[R]$. (5.6) and (5.7) allow us to find normalized blocks $u_1 < \cdots < u_n$, positive scalars $(\lambda_i)_{i=1}^n$ and $I \subset \{1,\dots,n\}$

such that

$$\begin{aligned}(5.8)\quad & \sum_{i=1}^{n} \lambda_i u_i \text{ is an } (\alpha+\beta)-\text{average supported by } Q,\\ & (u_i)_{i\in I} \text{ is } S_\beta-\text{admissible and } F\setminus\{\min F\}\subset\{\min \operatorname{supp} u_i : i\in I\},\\ & u_i \text{ is an } \alpha-\text{average for all } i\in I \text{ and } \|\sum_{i\in\{1,\dots,n\}\setminus I}\lambda_i u_i\|_{\ell_1}<\epsilon/2,\\ & |\lambda_i-\lambda_j|<\epsilon/2, \text{ for all } i,j \text{ in } I.\end{aligned}$$

Since $\sum_{i=1}^{n}\lambda_i u_i$ is supported by $Q\subset Q_0$, and $[Q_0]\cap\mathcal{D}=\emptyset$, we must have that $\max_{i\in I}\lambda_i\geq\epsilon$. We deduce from (5.8) that

$$\lambda_i\geq\epsilon/2 \text{ for all } i\in I.$$

Set $J_0=\{i\in I : \min F<\min\operatorname{supp} u_i\}$ and note that (5.8) implies that $F\setminus\{\min F\}\subset\{\min\operatorname{supp} u_i : i\in J_0\}$. It follows now, since F is maximal in S_β, that $\{\min F\}\cup\{\min\operatorname{supp} u_i : i\in J_0\}$ contains a maximal member of S_β as a subset and therefore, as S_β is stable, there exists an initial segment J of J_0 such that $\{\min F\}\cup\{\min\operatorname{supp} u_i : i\in J\}$ is a maximal member of S_β. Note also that $\|\sum_{i\in J}\lambda_i u_i\|\leq 3$.

Summarizing, given $M\in[Q_1]$ we found a block basis of α-averages $v_1<\cdots<v_k$, supported by M, $m\in M$ with $m<\min\operatorname{supp} v_1$, and scalars $(\mu_i)_{i=1}^k$ in $[\epsilon/2,2]$ so that

$$(5.9)\quad \{m\}\cup\{\min\operatorname{supp} v_i : i\leq k\} \text{ is maximal in } S_\beta \text{ and } \|\sum_{i=1}^{k}\mu_i v_i\|\leq 3.$$

Define

$$\begin{aligned}\mathcal{D}_1=\big\{L\in[Q_1] : \exists\,(\mu_i)_{i=1}^k\subset[\epsilon/2,2],\ \|\sum_{i=1}^{k}\mu_i\alpha_i^{L\setminus\{\min L\}}\|\leq 3, \text{ and}\\ \{\min L\}\cup\{\min\operatorname{supp}\alpha_i^{L\setminus\{\min L\}} : i\leq k\} \text{ is maximal in } S_\beta\big\}.\end{aligned}$$

Lemma 4.3 and the stability of S_β yield that $\mathcal{D}_1$ is closed in the topology of pointwise convergence. We now infer from (5.9) that every $M\in[Q_1]$ contains some $L\in\mathcal{D}_1$ as a subset. Thus, we deduce from Theorem 4.5 that there exists $M_0\in[Q_1]$ with $[M_0]\subset\mathcal{D}_1$.

Now let $L\in[M_0]$ and denote by n_L the unique integer such that $(\alpha_i^L)_{i=1}^{n_L}$ is maximally S_β-admissible. Because $L\in\mathcal{D}_1$, we must have that

$$(5.10)\qquad \begin{aligned}&\|\sum_{i=1}^{n_L}\mu_i\alpha_i^L\|\leq 4, \text{ for some choice of scalars}\\ &(\mu_i)_{i=1}^{n_L} \text{ in the interval } [\epsilon/2,2].\end{aligned}$$

Set $g_i=\alpha_i^{M_0}$, for all $i\in\mathbb{N}$. Then (g_i) is a normalized weakly null sequence, as $\vec{s}$ is assumed to be shrinking. Theorem 5.10 now yields a constant $C>0$

and a subsequence of (g_i) (which, for clarity, is still denoted by (g_i)), such that

$$\|\sum_{i\in G} a_i g_i\| \leq C\|\sum_{i=1}^{\infty} a_i g_i\|,$$

for every finitely supported scalar sequence (a_i) in $[-2,2]$ and every $G \subset \{i \in \mathbb{N} : |a_i| \geq \epsilon/2\}$. It follows from this, Lemma 4.3 and (5.10) that, whenever $F \in [\mathbb{N}]^{<\infty}$ is so that $(g_i)_{i\in F}$ is maximally S_β-admissible, then we have some choice of scalars $(\mu_i)_{i\in F}$ in $[\epsilon/2, 2]$ such that

$$\|\sum_{i\in F} \sigma_i \mu_i g_i\| \leq 8C,$$

for every choice of signs $(\sigma_i)_{i\in F}$. We conclude from the above, that some subsequence of (g_i) is a c_0^β-spreading model. Lemma 4.3 finally implies that there is some $L \in [M_0]$ (and thus $L \in [Q_1]$) such that (α_i^L) is a c_0^β-spreading model, contradicting the choice of Q_1. Therefore, we must have that $[Q] \subset \mathcal{D}$, for some $Q \in [P]$, and the proof of the theorem is now complete. □

6. Transfinite averages of weakly null sequences in $C(K)$ equivalent to the unit vector basis of c_0

In this section we present the following

Theorem 6.1. *Let K be a compact metric space and let (f_n) be a normalized, basic sequence in $C(K)$. Suppose that there exist $M \in [\mathbb{N}]$ and a summable sequence of positive scalars (ϵ_n) such that for all $t \in K$, the set $\{n \in M : |f_n(t)| \geq \epsilon_n\}$ is finite. Then there exist $\xi < \omega_1$ and a block basis of ξ-averages of (f_n) equivalent to the unit vector basis of c_0.*

(Note that all transfinite averages of (f_n) are considered with respect to $\mathcal{F} = S_1$.)

Remark 6.2. *The hypotheses in Theorem 6.1 imply that $\sum_{n\in M} |f_n(t)|$ is a convergent series, for all $t \in K$. It follows then from Rainwater's theorem* [36]*, that every normalized block basis of $(f_n)_{n\in M}$ is weakly null and therefore, the subsequence $(f_n)_{n\in M}$ of (f_n) is shrinking. Moreover, the convergence of the series $\sum_{n\in M} |f_n(t)|$ for all $t \in K$, implies that some block basis of $(f_n)_{n\in M}$ is equivalent to the unit vector basis of c_0. This is a special case of a famous result, due to J. Elton* [19]*, which states that if (x_n) is a normalized basic sequence in some Banach space and the series $\sum_n |x^*(x_n)|$ converges for every extreme point x^* in the ball of X^*, then some block basis of (x_n) is equivalent to the unit vector basis of c_0. An alternate proof of this special case of Elton's theorem is given in* [22]*. See also* [20], [4] *for related results. We wish to indicate however, as our next corollary shows, that this special case of Elton's theorem is also a consequence of Theorem 6.1. Hence, our result may be viewed as a quantitative version of this special case of Elton's theorem.*

Corollary 6.3. *Let (f_n) be a normalized basic sequence in $C(K)$ such that $\sum_n |f_n(t)|$ is a convergent series, for all $t \in K$. Then there exist $\xi < \omega_1$ and a block basis of ξ-averages of (f_n) equivalent to the unit vector basis of c_0.*

The proof is given at the end of this section.

The ordinal ξ that appears in the conclusion of Theorem 6.1, is related to the complexity of the compact family $\{F \in [M]^{<\infty} : \exists\, t \in K \text{ with } |f_n(t)| \geq \epsilon_n, \forall\, n \in F\}$. It follows from Corollary 3.2, that every normalized weakly null sequence in $C(K)$, for K a countable compact metric space, admits a subsequence satisfying the hypotheses of Theorem 6.1. Moreover, if K is homeomorphic to $[1, \omega^{\omega^\alpha}]$, for some $\alpha < \omega_1$, then as is shown in Corollary 6.8, the ordinal ξ in the conclusion of Theorem 6.1 can be taken not to exceed α.

We shall next describe how to obtain the "optimal" ξ satisfying the conclusion of Theorem 6.1.

The following conventions hold throughout this section. K is a compact metric space and $\vec{s} = (f_n)$ is a normalized shrinking basic sequence in $C(K)$. We shall assume, without loss of generality, by passing to a subsequence if necessary, that $\vec{s}$ is 2-basic. We let $\mathcal{F} = S_1$. *All transfinite averages of $\vec{s}$ will be taken with respect to $\mathcal{F}$.* As in the previous section, α_n^M abbreviates $\alpha_n^{\mathcal{F},\vec{s},M}$. In the sequel, when we refer to a block basis we shall always mean a block basis of $\vec{s} = (f_n)$. Also all transfinite averages will be taken with respect to $\vec{s}$.

Definition 6.4.
1. *Given $N \in [\mathbb{N}]$ and $1 \leq \alpha < \omega_1$, we say that N is α-large, if for every $\beta < \alpha$ and $M \in [N]$ there exists $L \in [M]$ such that no block basis of β-averages supported by L is a c_0^γ-spreading model, where $\beta + \gamma = \alpha$.*
2. *Given $N \in [\mathbb{N}]$ set $\xi^N = \sup\{\alpha < \omega_1 : \exists \text{ an } \alpha-\text{large}\, M \in [N]\}$. Put $\xi^N = 0$, if this set is empty. Finally put $\xi^0 = \min\{\xi^N : N \in [\mathbb{N}]\}$.*

Note that if $\xi^0 = \xi^{N_0}$ for some $N_0 \in [\mathbb{N}]$, then $\xi^L = \xi^0$, for all $L \in [N_0]$. In fact, if $1 \leq \xi^0 < \omega_1$, then every infinite subset of N_0 is ξ^0-large.

Proposition 6.5. *Suppose that $\xi^N < \omega_1$, for some $N \in [\mathbb{N}]$. Then there exists a block basis of ξ^N-averages, supported by N, which is equivalent to the unit vector basis of c_0.*

We postpone the proof and observe that if $\xi^0 < \omega_1$ and $\xi^0 = \xi^{N_0}$, then Proposition 6.5 yields that every infinite subset of N_0 supports a block basis of ξ^0-averages, equivalent to the unit vector basis of c_0 and, moreover, it follows by our preceding comments, that ξ^0 is the smallest ordinal with this property. Therefore, the optimality of ξ^0 is considered in this sense. In order to prove Theorem 6.1, we need to introduce some more notation and terminology.

Definition 6.6.
1. *Let $\beta < \alpha < \omega_1$, $p \in \mathbb{N}$ and $\epsilon > 0$. An α-average $u = \sum_i a_i f_i$, is said to be (β, p, ϵ)-large, if for every choice $I_1 <$*

$\cdots < I_k$ *of* k *consecutive members of* S_β, $k \le p$, *and all* $t \in K$, *we have* $|\sum_{i\in I} a_i f_i(t)| \le \epsilon + \sum_{i \notin I} a_i |f_i(t)|$, *where* $I = \cup_{j=1}^k I_j$.

(2) *Let* $N \in [\mathbb{N}]$, $1 \le \alpha < \omega_1$ *We say that* N *is* α*-nice if for every* $\beta < \alpha$, *every* $M \in [N]$, *every* $p \in \mathbb{N}$ *and all* $\epsilon > 0$, *there exists an* α*-average supported by* M *which is* (β, p, ϵ)*-large.*

The main step for proving Theorem 6.1 is

Theorem 6.7. *Suppose that* $N \in [\mathbb{N}]$ *is* α*-large, for some* $1 \le \alpha < \omega_1$. *Then* N *is* α*-nice.*

We postpone the proof in order to give the

Proof of Theorem 6.1. Let

$$\mathcal{G} = \{F \in [\mathbb{N}]^{<\infty} : \exists\, t \in K \text{ with } |f_n(t)| \ge \epsilon_n,\ \forall\, n \in F\}.$$

Clearly, $\mathcal{G}$ is hereditary. The compactness of K and our assumptions, imply that $\mathcal{G}[M]$ is compact in the topology of pointwise convergence. It follows that there is a countable ordinal ζ such that $\mathcal{G}[M]^{(\zeta)}$ is finite. Write $\zeta = \omega^\gamma k + \eta$, for some $k \in \mathbb{N}$ and $\eta < \omega^\gamma$. We infer now by the result of [21], that there exists $N \in [M]$ with the property $\mathcal{G}[N] \subset S_{\gamma+1}$.

We claim that $\xi^N \le \gamma + 1$ (see Definition 6.4). Indeed, were this claim false, we would choose $P \in [N]$ and a countable ordinal $\beta > \gamma + 1$ such that P is β-large. Theorem 6.7 then yields P is β-nice (see Definition 6.6). Next, let $\epsilon > 0$ and choose $Q \in [P]$ such that $\sum_{n \in Q} \epsilon_n < \epsilon/12$. Since $\gamma + 1 < \beta$ and P is β-nice, there exists a β-average $u = \sum_i a_i f_i$, supported by Q which is $(\gamma+1, 1, \epsilon/2)$-large. This means

$$\Big|\sum_{i \in I} a_i f_i(t)\Big| \le \epsilon/2 + \sum_{i \notin I} a_i |f_i(t)|,$$

for all $t \in K$ and every $I \in S_{\gamma+1}$. Given $t \in K$, put $\Lambda_t = \{n \in \mathbb{N} : |f_n(t)| \ge \epsilon_n\}$. Note that u is supported by N and so $\Lambda_t \cap \operatorname{supp} u \in S_{\gamma+1}$, for all $t \in K$, as $\Lambda_t \cap \operatorname{supp} u \in \mathcal{G}[N]$. Taking in account that $\|u\| = 1$, we have $0 \le a_i \le 3$, for all $i \in \mathbb{N}$. Hence,

$$\begin{aligned} |u(t)| &\le \left|\sum_{i \in \Lambda_t \cap \operatorname{supp} u} a_i f_i(t)\right| + \left|\sum_{i \notin \Lambda_t} a_i f_i(t)\right| \\ &\le \epsilon/2 + 2 \sum_{i \notin \Lambda_t} a_i |f_i(t)| \\ &< \epsilon/2 + 6\epsilon/12 = \epsilon, \end{aligned}$$

for all $t \in K$. Since ϵ was arbitrary, we have reached a contradiction. Therefore, our claim holds. In particular, $\xi^N < \omega_1$ and the assertion of the theorem is a consequence of Proposition 6.5. □

Corollary 6.8. *Let* (f_n) *be a normalized weakly null sequence in* $C(\omega^{\omega^\xi})$, $\xi < \omega_1$. *Then there exist* $\alpha \le \xi$ *and a block basis of* α*-averages of* (f_n) *equivalent to the unit vector basis of* c_0.

Proof. Set $K = [1, \omega^{\omega^\xi}]$. Corollary 3.2 yields $M \in [\mathbb{N}]$ and a summable sequence of positive scalars (ϵ_n) such that for all $t \in K$ the set $\{n \in M : |f_n(t)| \geq \epsilon_n\}$ belongs to S_ξ^+. In particular, $\Lambda_t \cap M$ is the union of two consecutive members of S_ξ. The argument in the proof of Theorem 6.1 shows that $\xi^M \leq \xi$. The assertion of the corollary now follows from Proposition 6.5. $\square$

We shall now give the proof of Proposition 6.5. This requires two lemmas.

Lemma 6.9. *Suppose that $1 \leq \alpha < \omega_1$. Let $m < n$ in $\mathbb{N}$ and $F \in [\mathbb{N}]^{<\infty}$ with $n < \min F$ be such that $\{n\} \cup F$ is a maximal member of S_α. Then $\{m\} \cup F \notin S_\alpha$.*

Proof. We use transfinite induction on α. When $\alpha = 1$, we must have that $|F| = n-1$, in order for $\{n\} \cup F$ be maximal in S_1. Hence, $|\{m\} \cup F| = n > m = \min(\{m\} \cup F)$. Thus the assertion of the lemma holds in this case.

Next assume the assertion holds for all ordinals smaller than α ($\alpha > 1$). Suppose first α is a limit ordinal and let (α_n) be the sequence of successor ordinals associated to α. Since $\{n\} \cup F$ is maximal in S_α, we have that $\{n\} \cup F$ is maximal in S_{α_k}, for all $k \leq n$ such that $\{n\} \cup F \in S_{\alpha_k}$. Suppose we had $\{m\} \cup F \in S_\alpha$. Then there is some $k \leq m$ such that $\{m\} \cup F \in S_{\alpha_k}$. We infer from the spreading property of S_{α_k}, as $m < n$, that $\{n\} \cup F \in S_{\alpha_k}$. Therefore, $\{n\} \cup F$ is maximal in S_{α_k}. The induction hypothesis applied on α_k now yields $\{m\} \cup F \notin S_{\alpha_k}$, a contradiction which proves the assertion when α is a limit ordinal.

We now assume α is a successor ordinal, say $\alpha = \beta + 1$. Since $\{n\} \cup F$ is maximal in S_α, there exist $F_1 < \cdots < F_n$, successive maximal members of S_β such that $\{n\} \cup F = \cup_{i=1}^n F_i$ (see [21]). We shall assume $m > 1$ or else the assertion holds since $\{1\}$ is maximal in every Schreier family and $F \neq \emptyset$. Note that the induction hypothesis on β implies that $G_1 = \{m\} \cup (F_1 \setminus \{n\}) \notin S_\beta$. It follows, as S_β is stable, that G_1 contains a maximal member H_1 of S_β as an initial segment, and so we may write $G_1 = H_1 \cup H_2$ with $H_2 \neq \emptyset$. Of course, $m = \min H_1$. Set $H = H_1 \cup \cup_{i=2}^m F_i$. Then H is maximal in S_α. This completes the proof of the lemma since H is a proper subset of $\{m\} \cup F$. $\square$

Lemma 6.10. *Let $P \in [\mathbb{N}]$, $\beta \leq \alpha < \omega_1$ and $\tau < \omega_1$. Assume that every block basis of β-averages supported by P is a c_0^γ-spreading model, where $\beta + \gamma = \alpha$, while every block basis of α-averages supported by P is a c_0^τ-spreading model. Then there exists $Q \in [P]$ such that every block basis of β-averages supported by Q is a $c_0^{\gamma+\tau}$-spreading model.*

Proof. We assume that both γ and τ are greater than or equal to 1, or else the assertion of the lemma is trivial. We also assume, without loss of generality thanks to Lemma 4.6, that there exists a constant $C > 0$ such that every block basis of β-averages (resp. α-averages) supported by P is a c_0^γ (resp. c_0^τ)-spreading model with constant C. We shall further assume, without loss of generality thanks to Lemma 5.8, that for every $F \in S_{\gamma+\tau}[P]$ we have $F \setminus \{\min F\} \in S_\tau[S_\gamma]$.

Let $M \in [P]$. Choose a sequence of positive scalars (δ_i) with $\sum_i \delta_i < 1/(4C)$. We apply Proposition 5.9, successively, to obtain the following objects:

(1) A maximally S_τ-admissible block basis $v_1 < \cdots < v_n$ of α-averages, supported by M, with $\min M < \min \operatorname{supp} v_1$.
(2) Successive, maximal members $F_1 < \cdots < F_n$ of $S_\gamma[M]$ such that $\max \operatorname{supp} v_i < \min F_{i+1}$, for all $i < n$.
(3) Successive finite subsets of $\mathbb{N}$ $J_1 < \cdots < J_n$ such that for each $i \leq n$, there exist a normalized block basis $(u_j)_{j \in J_i}$, a subset I_i of J_i and positive scalars $(\lambda_j)_{j \in J_i}$ which satisfy the following properties:

$$v_i = \sum_{j \in J_i} \lambda_j u_j, \text{ and } \Big\| \sum_{j \in J_i \setminus I_i} \lambda_j u_j \Big\|_{\ell_1} < \delta_i. \tag{6.1}$$

$$\begin{aligned} &(u_j)_{j \in I_i} \text{ is an } S_\gamma - \text{admissible block basis of } \beta - \text{averages} \\ &\text{and } |\lambda_r - \lambda_s| < \delta_i, \text{ for all } r, s \text{ in } I_i. \end{aligned} \tag{6.2}$$

$$F_i \setminus \{\min F_i\} \subset \{\min \operatorname{supp} u_j : j \in I_i\}. \tag{6.3}$$

Our assumptions yield that $\| \sum_{i=1}^n v_i \| \leq C$ and that

$$1 - \delta_i \leq \Big\| \sum_{j \in I_i} \lambda_j u_j \Big\| \leq C \max_{j \in I_i} \lambda_j, \text{ for all } i \leq n.$$

(6.2) now implies

$$1/(2C) \leq \lambda_j \leq 3, \text{ for all } j \in I_i \text{ and } i \leq n. \tag{6.4}$$

We also obtain from (6.1) that

$$\Big\| \sum_{i=1}^n \sum_{j \in I_i} \lambda_j u_j \Big\| \leq C + \sum_{i=1}^n \delta_i < 2C. \tag{6.5}$$

We next observe that for all $i < n$ and $j_0 \in I_i$, $\{\min \operatorname{supp} u_{j_0}\} \cup \{\min \operatorname{supp} u_j : j \in I_{i+1}\} \notin S_\gamma$. This is so since $F_{i+1} \setminus \{\min F_{i+1}\} \subset \{\min \operatorname{supp} u_j : j \in I_{i+1}\}$, (by (6.3)), $\max \operatorname{supp} v_i < \min F_{i+1}$, and thus, as a consequence of Lemma 6.9, we have that $\{\min \operatorname{supp} u_{j_0}\} \cup (F_{i+1} \setminus \{\min F_{i+1}\}) \notin S_\gamma$.

It follows from this that for all $i \leq n$ there exists an initial segment I_i^* of I_i (possibly, $I_i^* = \emptyset$) with $\max I_i^* < \max I_i$, such that $\{\min \operatorname{supp} u_j : j \in (I_i \setminus I_i^*) \cup I_{i+1}^*\}$ is a maximal member of S_γ, for all $i < n$. Note that $I_1^* = \emptyset$.

Set $T_i = (I_i \setminus I_i^*) \cup I_{i+1}^*$, for all $i < n$. Then $(u_j)_{j \in T_i}$ is maximally S_γ-admissible for all $i < n$. We also infer from (6.4) and (6.5) that

$$\Big\| \sum_{j \in \cup_{i<n} T_i} \lambda_j u_j \Big\| \leq 4C, \, \lambda_j \in [1/(2C), 3], \text{ for all } j \in \cup_{i<n} T_i.$$

Note also that $\min \operatorname{supp} u_{\min T_i} < \min \operatorname{supp} v_{i+1}$, for all $i < n$. Since $\min M < \min \operatorname{supp} v_1$ and $(v_i)_{i=1}^n$ is maximally S_τ-admissible, Lemma 6.9 and the

spreading property of S_τ, yield that $\{\min M\} \cup \{\min \operatorname{supp} u_{\min T_i} : i < n\}$ is not a member of S_τ. Hence, by the stability of S_τ, there exists $m < n$ such that $\{\min M\} \cup \{\min \operatorname{supp} u_{\min T_i} : i \leq m\}$ is a maximal member of S_τ. Note also that $\| \sum_{j \in \cup_{i \leq m} T_i} \lambda_j u_j \| \leq 4C$ and $\lambda_j \in [1/(2C), 3]$, for all $j \in \cup_{i \leq m} T_i$.

Summarizing, given $M \in [P]$ there exists a maximally $S_\tau[S_\gamma]$-admissible block basis $(u_i)_{i=1}^k$ of β-averages, supported by M, and scalars $(\lambda_i)_{i=1}^k$ in $[1/(2C), 3]$ such that $\| \sum_{i=1}^k \lambda_i u_i \| \leq 5C$. Given $L \in [P]$ let n_L denote the unique integer such that $(\beta_i^L)_{i=1}^{n_L}$ is maximally $S_\tau[S_\gamma]$-admissible. Define

$$\mathcal{D} = \left\{ L \in [P] : \exists\, (\lambda_i)_{i=1}^{n_L} \subset [1/(2C), 3],\ \Big\| \sum_{i=1}^{n_L} \lambda_i \beta_i^L \Big\| \leq 5C \right\}.$$

Lemma 4.3 and the stability of $S_\tau[S_\gamma]$ yield that $\mathcal{D}$ is closed in the topology of pointwise convergence. We infer from our preceding discussion, that every $M \in [P]$ contains some $L \in \mathcal{D}$ as a subset. Thus, we deduce from Theorem 4.5 that there exists $M_0 \in [P]$ with $[M_0] \subset \mathcal{D}$. Arguing as in the last part of the proof of Theorem 5.1, using Theorem 5.10 and our assumptions on P, we obtain a block basis of β-averages which is a $c_0^{\gamma+\tau}$-spreading model. The assertion of the lemma now follows from Lemma 4.6. □

Proof of Proposition 6.5. To simplify our notation, let us write ξ instead of ξ^N. We assert that for every $M \in [N]$ and all $\beta < \omega_1$ there exists a block basis of ξ-averages supported by M which is a c_0^β-spreading model. Once this is accomplished, the proposition will follow from the Kunen-Martin boundedness principle (see [16], [25]). To see this, let $N \in [\mathbb{N}]$. Given $n \in \mathbb{N}$, let $\mathcal{T}_n^N$ denote the family of those finite subsets of N that are initial segments of sets of the form $\cup_{i=1}^k \operatorname{supp} \xi_i^L$, for some $k \in \mathbb{N}$ and $L \in [N]$ such that $\| \sum_{i=1}^k \xi_i^L \| \leq n$. We claim there is some $n \in \mathbb{N}$ so that $\mathcal{T}_n^N$ is not compact in the topology of pointwise convergence. Otherwise, the Mazurkiewicz-Sierpinski theorem [29], yields $\zeta < \omega_1$ so that $\mathcal{T}_n^N$ is homeomorphic to a subset of $[1, \omega^{\omega^\zeta}]$, for all $n \in \mathbb{N}$. We may now choose, according to our assertion combined with Lemma 4.6, some $L_0 \in [N]$ and $n \in \mathbb{N}$ such that $(\xi_i^L)_{i=1}^\infty$ is a $c_0^{\zeta+1}$-spreading model with constant n, for all $L \in [L_0]$. It follows from this that for all $L \in [L_0]$, $\cup_{i=1}^{n_L} \operatorname{supp} \xi_i^L \in \mathcal{T}_n^N$, where n_L stands for the unique integer such that $(\xi_i^L)_{i=1}^{n_l}$ is maximally $S_{\zeta+1}$-admissible. Since S_α is homeomorphic to $[1, \omega^{\omega^\alpha}]$ for all $\alpha < \omega_1$ (see [1]), this implies that $S_{\zeta+1}$ is homeomorphic to a subset of $[1, \omega^{\omega^\zeta}]$ which is absurd. Hence, indeed, there is some $n \in \mathbb{N}$ with $\mathcal{T}_n^N$ non-compact. Subsequently, there exists $M \in [N]$, $M = (m_i)$, such that $\{m_1, \dots, m_k\} \in \mathcal{T}_n^N$, for all $k \in \mathbb{N}$. We now infer from Lemma 4.3, that $\| \sum_{i=1}^k \xi_i^M \| \leq n$, for all $k \in \mathbb{N}$. Using an argument based on Theorem 4.5, similar to that in the proof of Lemma 4.6, we conclude that some block basis of ξ-averages is equivalent to the unit vector basis of c_0.

We shall next prove our initial assertion by transfinite induction on β. The assertion is trivial for $\beta = 0$. Assume $\beta \geq 1$ and that the assertion

holds for all $M \in [N]$ and all ordinals smaller than β yet, for some $P \in [N]$ there is no block basis of ξ-averages, supported by P, which is a c_0^β-spreading model. We now show that P is $(\xi+\beta)$-large which, of course, is absurd.

To see this, first consider an ordinal $\gamma < \xi$ and let $M \in [P]$. Write $\xi = \gamma + \delta$. We claim that there exists $L \in [M]$ such that no block basis of γ-averages supported by L is a $c_0^{\delta+\beta}$-spreading model (note that $\gamma+(\delta+\beta) = \xi+\beta$). Were this claim false, then Lemma 4.6 would yield a constant $C > 0$ and $L_0 \in [M]$ such that, every block basis of γ-averages supported by L_0 is a $c_0^{\delta+\beta}$-spreading model with constant C. By employing Lemma 5.8 we may assume, without loss of generality, that for all $F \in S_\beta[S_\delta]$, $F \subset L_0$, we have $F \setminus \{\min F\} \in S_{\delta+\beta}$. But now, we shall exhibit a block basis of ξ-averages supported by L_0 (and thus also by P), which is a c_0^β-spreading model. Indeed, as $\xi = \gamma + \delta$, we may apply Proposition 5.9, successively, to obtain block bases $u_1 < u_2 < \dots$ and $v_1 < v_2 < \dots$ consisting of ξ and γ-averages, respectively, both supported by L_0; A sequence of positive scalars (λ_i) and a sequence $F_1 < F_2 < \dots$ of successive finite subsets of $\mathbb{N}$ so that the following requirements are satisfied:

(1) $\|u_i - \sum_{j \in F_i} \lambda_j v_j\| < \epsilon_i$, for all $i \in \mathbb{N}$.
(2) $(v_j)_{j \in F_i}$ is S_δ-admissible and $\operatorname{supp} v_j \subset \operatorname{supp} u_i$, for all $j \in F_i$ and $i \in \mathbb{N}$.

In the above, (ϵ_i) is a summable sequence of positive scalars. Since $(\lambda_j)_{j \in \cup_i F_i}$ is bounded and (v_i) is a $c_0^{\delta+\beta}$-spreading model, our assumptions on L_0 readily imply that (u_i) is a block basis of ξ-averages supported by P which is a c_0^β-spreading model. This contradicts the choice of P. Therefore our claim holds.

Next, let $M \in [P]$, $\gamma < \beta$ and write $\beta = \gamma+\delta$. Note that $\xi+\beta = (\xi+\gamma)+\delta$. We now claim that there exists $L \in [M]$ such that no block basis of $(\xi+\gamma)$-averages supported by L is a c_0^δ-spreading model. If that were not the case then, thanks to Lemma 4.6, there would exist $L_0 \in [M]$ such that every block basis of $(\xi+\gamma)$-averages supported by L_0 is a c_0^δ-spreading model.

Since $\gamma < \beta$, the induction hypothesis combined with Lemma 4.6 implies the existence of some $L_1 \in [L_0]$ such that every block basis of ξ-averages supported by L_1 is a c_0^γ-spreading model. We deduce from Lemma 6.10 that some block basis of ξ-averages supported by L_0 (and thus also by P) is a $c_0^{\gamma+\delta}$-spreading model. Since $\beta = \gamma+\delta$, we contradict the choice of P. Therefore, this claim holds as well.

Summarizing, we showed that for every $\gamma < \xi+\beta$ and all $M \in [P]$ there exists $L \in [M]$ such that no block basis of γ-averages supported by L is a c_0^δ-spreading model, where $\gamma+\delta = \xi+\beta$. But this means $P \in [N]$ is $(\xi+\beta)$-large, contradicting the definition of ξ. The proof of the proposition is now complete. $\square$

In the next part of this section we give the proof of Theorem 6.7. We shall need a few technical lemmas.

Lemma 6.11. *Suppose that $N \in [\mathbb{N}]$ is α-nice (see Definition 6.6). Then for every $P \in [N]$, every $\beta < \alpha$, every $p \in \mathbb{N}$ and all $\epsilon > 0$, there exists $M \in [P]$ such that every α-average supported by M is (β, p, ϵ)-large.*

Proof. Define $\mathcal{D} = \{L \in [P] : \alpha_1^L \text{ is } (\beta, p, \epsilon) - \text{large}\}$. Lemma 4.3 yields $\mathcal{D}$ is closed in the topology of pointwise convergence. Because N is α-nice, we deduce that $[L] \cap \mathcal{D} \neq \emptyset$, for all $L \in [P]$. We infer now, from Theorem 4.5, that $[M] \subset \mathcal{D}$, for some $M \in [P]$. Clearly, M is as desired. □

Lemma 6.12. *Suppose that $N_1 \supset N_2 \supset \dots$ are infinite subsets of $\mathbb{N}$ and $\alpha_1 < \alpha_2 < \dots$ are countable ordinals such that N_i is α_i-nice for all $i \in \mathbb{N}$. Let $N \in [\mathbb{N}]$ be such that $N \setminus N_i$ is finite, for all $i \in \mathbb{N}$. Then, N is α-nice, where $\alpha = \lim_i \alpha_i$.*

Proof. Let $M \in [N]$, $\beta < \alpha$, $p \in \mathbb{N}$ and $\epsilon > 0$. It suffices to find an α-average u supported by M which is (β, p, ϵ)-large. Choose a sequence of positive scalars (δ_i) with $\sum_i \delta_i < \epsilon/6$.

Let $k \in \mathbb{N}$ be such that $\beta < \alpha_k$. Since N_k is α_k-nice, we may apply Lemma 6.11, successively, to obtain infinite subsets $P_1 \supset P_2 \supset \dots$ of $M \cap N_k$ such that, for all $i \in \mathbb{N}$, every α_k-average supported by P_i is (β, p, δ_i)-large. Next choose integers $p_1 < p_2 < \dots$ such that $p_i \in P_i$, for all $i \in \mathbb{N}$, and set $P = (p_i)$.

We now employ Proposition 5.9 to find $Q \in [P]$ with the property that every α-average supported by Q admits an $(\epsilon/2, \alpha_k, \beta_k)$-decomposition (see Definition 5.2), where $\alpha_k + \beta_k = \alpha$. Let u be an α-average supported by Q. Write $u = \sum_{i=1}^n \lambda_i u_i$, where $u_1 < \dots < u_n$ are normalized blocks, $(\lambda_i)_{i=1}^n$ are positive scalars for which there exists $I \subset \{1, \dots, n\}$ satisfying

$$u_i \text{ is an } \alpha_k - \text{average for all } i \in I, \text{ while } \Big\| \sum_{i \in \{1,\dots,n\} \setminus I} \lambda_i u_i \Big\|_{\ell_1} < \epsilon/2.$$

If $u_i = \sum_s a_s^i f_s$, for $i \leq n$, then, clearly, $\sum_{i \in \{1,\dots,n\} \setminus I} \lambda_i \sum_s a_s^i < \epsilon/2$.

We are going to show that u is (β, p, ϵ)-large. To this end, let J be the union of less than, or equal to, p consecutive members of S_β and let $t \in K$. Write $I = \{i_1 < \dots, < i_m\}$. Observe that u_{i_j} is an α_k-average supported by P_j and thus by the choice of P_j,

$$\Big|\sum_{s \in J} a_s^{i_j} f_s(t)\Big| \leq \delta_j + \sum_{s \notin J} a_s^{i_j} |f_s(t)|, \text{ for all } j \leq m.$$

Therefore, letting $I^c = \{1, \dots, n\} \setminus I$,

$$\begin{aligned}
\Big|\sum_{i=1}^n \lambda_i \sum_{s \in J} a_s^i f_s(t)\Big| &\leq \Big|\sum_{i \in I^c} \lambda_i \sum_{s \in J} a_s^i f_s(t)\Big| + \Big|\sum_{i \in I} \lambda_i \sum_{s \in J} a_s^i f_s(t)\Big| \\
&\leq \sum_{i \in I^c} \lambda_i \sum_s a_s^i + \sum_{i \in I} \lambda_i \Big|\sum_{s \in J} a_s^i f_s(t)\Big| \\
&\leq \epsilon/2 + \sum_{j=1}^m \lambda_{i_j} \big(\delta_j + \sum_{s \notin J} a_s^{i_j} |f_s(t)|\big)
\end{aligned}$$

$$\leq \epsilon/2 + 3\sum_{j=1}^{|I|}\delta_j + \sum_{i=1}^{n}\lambda_i\sum_{s\notin J}a_s^i|f_s(t)|$$
$$\leq \epsilon + \sum_{i=1}^{n}\lambda_i\sum_{s\notin J}a_s^i|f_s(t)|.$$

The proof of the lemma is now complete. □

Lemma 6.13. *Let $u_1 < \cdots < u_n$ be a normalized finite block basis of (f_i). Write $u_i = \sum_s a_s^i f_s$, and set $k_i = \max\operatorname{supp} u_i$ for all $i \leq n$. Let $\alpha < \omega_1$ and denote by $(\alpha_j + 1)_{j=1}^{\infty}$ the sequence of ordinals associated to α. Let $\mathcal{G}$ be a hereditary and spreading family, and $(\delta_i)_{i=1}^{n}$ be a sequence of non-negative scalars. Suppose that $J \in \mathcal{G}[S_\alpha]$ satisfies the following property: If $2 \leq i \leq n$ is so that $J \cap \operatorname{supp} u_i$ is contained in the union of less than, or equal to, k_{i-1} consecutive members of S_{α_j}, for some $j \leq k_{i-1}$ then,*

$$\Big|\sum_{s\in J}a_s^i f_s(t)\Big| \leq \delta_i + \sum_{s\notin J}|a_s^i||f_s(t)|, \text{ for all } t \in K.$$

Then for every scalar sequence $(b_i)_{i=1}^{n}$ and all $t \in K$, we have the estimate

(6.6)
$$\Big|\sum_{i=1}^{n}b_i\sum_{s\in J}a_s^i f_s(t)\Big| \leq \max\Big\{\Big|\sum_{i\in I}b_i u_i(t)\Big| : (u_i)_{i\in I} \text{ is } \mathcal{G}^+ - \text{admissible}\Big\}$$
$$+ \Big(\sum_{i=1}^{n}\delta_i\Big)\max_{i\leq n}|b_i| + \sum_{i=1}^{n}|b_i|\sum_{s\notin J}|a_s^i||f_s(t)|.$$

Proof. We may assume that $J \cap \cup_{i=1}^{n}\operatorname{supp} u_i \neq \emptyset$, or else the assertion of the lemma is trivial. We may thus write $J \cap \cup_{i=1}^{n}\operatorname{supp} u_i = \cup_{l=1}^{p} J_l$, where $J_1 < \cdots < J_p$ are non-empty members of S_α with $\{\min J_l : l \leq p\} \in \mathcal{G}$.

Define $I_l = \{i \leq n : r(u_i) \cap J_l \neq \emptyset\}$ (where $r(u_i)$ denotes the range of u_i) and $i_l = \min I_l$, for all $l \leq p$. Put $I = \{i_l : l \leq p\}$ and let I^c be the complement of I in $\{1,\ldots,n\}$. Then $(u_i)_{i\in I}$ is $\mathcal{G}^+$-admissible.

Indeed, set $L_i = \{l \leq p : i_l = i\}$, for all $i \in I$. Observe that L_i is an interval and that $L_i < L_{i'}$ for all $i < i'$ in I. Hence, $\min J_{\min L_i} \leq \max\operatorname{supp} u_i$, for all $i \in I$. Since $\mathcal{G}$ is hereditary and spreading, we infer that $(k_i)_{i\in I} \in \mathcal{G}$. It follows now, by the spreading property of $\mathcal{G}$, that $(u_i)_{i\in I\setminus\{\min I\}}$ is $\mathcal{G}$-admissible.

Next assume that $i \in I^c \cap \cup_{l\leq p} I_l$. Then there is a unique $l \leq p$ with $i \in I_l$. Otherwise, $r(u_i) \cap J_l \neq \emptyset$ for at least two distinct l's, and so $i \in I$.

It follows now that $J \cap \operatorname{supp} u_i = J_l \cap \operatorname{supp} u_i$, for some $l \leq p$. Note that $i_l < i$ and that $J_l \cap r(u_{i_l}) \neq \emptyset$. Therefore $\min J_l \leq k_{i_l}$. We deduce from this that $J_l \in S_{\alpha_j+1}$ for some $j \leq k_{i_l}$ and, subsequently, that J_l is contained in the union of less than or equal to k_{i_l} consecutive members of S_{α_j}, for some $j \leq k_{i_l}$. The same holds for $J \cap \operatorname{supp} u_i$ and as $i_l < i$, we infer from our

hypothesis, that

$$\Big|\sum_{s\in J} a_s^i f_s(t)\Big| \le \delta_i + \sum_{s\notin J} |a_s^i||f_s(t)|, \text{ for all } i \in I^c \text{ and } t \in K.$$

Now let $(b_i)_{i=1}^n$ be any scalar sequence and let $t \in K$. Then

$$\sum_{i=1}^n b_i \sum_{s\in J} a_s^i f_s(t) = \sum_{i\in I} b_i \sum_{s\in J} a_s^i f_s(t) + \sum_{i\in I^c} b_i \sum_{s\in J} a_s^i f_s(t).$$

Our preceding discussions yield

$$\begin{aligned} \Big|\sum_{i\in I^c} b_i \sum_{s\in J} a_s^i f_s(t)\Big| &\le \sum_{i\in I^c} |b_i|\Big|\sum_{s\in J} a_s^i f_s(t)\Big| \\ &\le \sum_{i\in I^c} |b_i|\big(\delta_i + \sum_{s\notin J} |a_s^i||f_s(t)|\big) \\ &\le (\max_{i\le n} |b_i|)\sum_{i\in I^c}\delta_i + \sum_{i\in I^c}|b_i|\sum_{s\notin J}|a_s^i||f_s(t)| \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} \Big|\sum_{i\in I} b_i \sum_{s\in J} a_s^i f_s(t)\Big| &= \Big|\sum_{i\in I} b_i\big(u_i(t) - \sum_{s\notin J} a_s^i f_s(t)\big)\Big| \\ &\le \Big|\sum_{i\in I} b_i u_i(t)\Big| + \sum_{i\in I}|b_i|\sum_{s\notin J}|a_s^i||f_s(t)|. \end{aligned} \tag{6.8}$$

Combining (6.7) with (6.8) we obtain (6.6), since $(u_i)_{i\in I}$ is $\mathcal{G}^+$-admissible. □

Lemma 6.14. *Suppose that $N \in [\mathbb{N}]$ is α-nice and that there exist $\Gamma \in [N]$ and $\gamma < \omega_1$ such that no block basis of α-averages supported by Γ is a c_0^γ-spreading model. Then there exist $M \in [N]$ and $1 \le \beta \le \gamma$ such that M is $(\alpha+\beta)$-nice.*

Proof. Define

$$\beta = \min\{\psi < \omega_1 : \exists\, \Psi \in [N] \text{ such that no block basis of } \alpha - \text{averages supported by } \Psi \text{ is a } c_0^\psi - \text{spreading model}\}.$$

Our assumptions yield $1 \le \beta \le \gamma$. Choose $M \in [N]$ such that no block basis of α-averages supported by M is a c_0^β-spreading model. We are going to show that M is $(\alpha+\beta)$-nice. Let $M_0 \in [M]$ and $\tau < \alpha+\beta$. Let $p \in \mathbb{N}$ and $\epsilon > 0$. We shall exhibit an $(\alpha+\beta)$-average supported by M_0 which is (τ, p, ϵ)-large. Choose a decreasing sequence of positive scalars (δ_i) such that $\sum_i \delta_i < \epsilon/6$.

We first consider the case $\tau < \alpha$. Because N is α-nice, we may apply Lemma 6.11, successively, to obtain infinite subsets $P_1 \supset P_2 \supset \dots$ of M_0 such that, for all $i \in \mathbb{N}$, every α-average supported by P_i is (τ, p, δ_i)-large. Choose integers $p_1 < p_2 < \dots$ such that $p_i \in P_i$, for all $i \in \mathbb{N}$, and set $P_0 = (p_i)$. Proposition 5.9 now yields an $(\alpha+\beta)$-average u supported

by P_0 and admitting an $(\epsilon/2, \alpha, \beta)$-decomposition (see Definition 5.2). In particular, there exist normalized blocks $u_1 < \cdots < u_n$, positive scalars $(\lambda_i)_{i=1}^n$ and $I \subset \{1, \dots, n\}$ such that $u = \sum_{i=1}^n \lambda_i u_i$, u_i is an α-average for all $i \in I$ and $\| \sum_{i \in \{1,\dots,n\} \setminus I} \lambda_i u_i \|_{\ell_1} < \epsilon/2$. Let J be the union of less than, or equal to, p consecutive members of S_τ, and let $t \in K$. By repeating the argument in the last part of the proof of Lemma 6.12 we conclude that u is (τ, p, ϵ)-large. This proves the assertion when $\tau < \alpha$.

Next suppose $\alpha \leq \tau < \alpha + \beta$ and choose $\zeta < \beta$ with $\tau = \alpha + \zeta$. Recall that the definition of β implies that every infinite subset of M_0 supports a block basis of α-averages which is a c_0^ζ-spreading model. Hence, thanks to Lemma 4.6, there will be no loss of generality in assuming that for some positive constant C, every block basis of α-averages supported by M_0 is a c_0^ζ-spreading model with constant C. We shall further assume, because of Lemma 5.8, that for every $F \in S_\tau[M_0]$ we have $F \setminus \{\min F\} \in S_\zeta[S_\alpha]$.

Let $(\alpha_j + 1)$ be the sequence of ordinals associated to α. We shall construct $m_1 < m_2 < \dots$ in M_0 with the following property: If $n \in \mathbb{N}$ and $j \leq m_n$, then every α-average supported by $\{m_i : i > n\}$ is $(\alpha_j, m_n, \delta_n)$-large. This construction is done inductively as follows: Choose $m_1 \in M_0$. Apply Lemma 5.6 to find $L_1 \in [M_1]$ with $m_1 < \min L_1$ and such that $S_{\alpha_j}[L_1] \subset S_{\alpha_{m_1}}$ for all $j \leq m_1$. We then employ Lemma 6.11, as N is α-nice, to obtain $M_1 \in [L_1]$ such that every α-average supported by M_1 is $(\alpha_{m_1}, m_1, \delta_1)$-large. It follows that every α-average supported by M_1 is $(\alpha_j, m_1, \delta_1)$-large, for all $j \leq m_1$. Set $m_2 = \min M_1$.

Suppose $n \geq 2$ and that we have selected integers $m_1 < \cdots < m_n$ in M_0, and infinite subsets $M_1 \supset \cdots \supset M_{n-1}$ of M_0 with $m_{i+1} = \min M_i$ and such that every α-average supported by M_i is $(\alpha_j, m_i, \delta_i)$-large for all $j \leq m_i$ and $i < n$.

We next choose, by Lemma 5.6, $L_n \in [M_{n-1}]$ with $m_n < \min L_n$ and such that $S_{\alpha_j}[L_n] \subset S_{\alpha_{m_n}}$, for all $j \leq m_n$. Because N is α-nice, Lemma 6.11 allows us select $M_n \in [L_n]$ such that every α-average supported by M_n is $(\alpha_j, m_n, \delta_n)$-large for all $j \leq m_n$. Set $m_{n+1} = \min M_n$. This completes the inductive step. Evidently, $m_1 < m_2 < \dots$ satisfy the required property.

We set $P = (m_n)$. The preceding construction yields the following fact that will be used later in the course of the proof: Suppose v is an α-average supported by P and $\min \operatorname{supp} v = m_n$, for some $n \geq 2$, then v is $(\alpha_j, m_{n-1}, \delta_{n-1})$-large, for all $j \leq m_{n-1}$.

Recall that no block basis of α-averages supported by P is a c_0^β-spreading model. Let $0 < \delta < \epsilon/(p(C+1)+3)$ and apply Theorem 5.1 to find an $(\alpha + \beta)$-average u supported by P, normalized blocks $u_1 < \cdots < u_n$, positive scalars $(\lambda_i)_{i=1}^n$ and $I \subset \{1, \dots, n\}$ such that $u = \sum_{i=1}^n \lambda_i u_i$, u_i is an α-average for all $i \in I$, $\| \sum_{i \in \{1,\dots,n\} \setminus I} \lambda_i u_i \|_{\ell_1} < \delta$ and $\max_{i \in I} \lambda_i < \delta$. We show u is (τ, p, ϵ)-large which will finish the proof of the lemma. Set

$$\mathcal{G} = \{F \in [\mathbb{N}]^{<\infty} : \exists F_1 < \cdots < F_p \text{ in } S_\zeta^+,\ F \subset \cup_{i=1}^p F_i\}.$$

$\mathcal{G}$ is a hereditary and spreading family.

Let $J \subset M_0$ be the union of less than, or equal to, p consecutive members of S_τ, and let $t \in K$. Our assumptions on M_0 yield $J \in \mathcal{G}[S_\alpha]$. Let $\{i_1 < \ldots, < i_m\}$ be an enumeration of I and put $m_{d_k} = \max \operatorname{supp} u_{i_k}$, for all $k \leq m$. It has been already remarked that u_{i_k} is $(\alpha_j, m_{d_{k-1}}, \delta_{d_{k-1}})$-large, for all $2 \leq k \leq m$ and $j \leq m_{d_{k-1}}$. It follows that the hypotheses of Lemma 6.13 are fulfilled for the block basis $u_{i_1} < \cdots < u_{i_m}$ and the given $J \subset M_0$, with "δ_1"$= 0$ and "δ_k"$= \delta_{d_{k-1}}$ for $2 \leq k \leq m$. Writing $u_i = \sum_s a_s^i f_s$, for all $i \leq n$, we infer from (6.6) that

$$\begin{aligned}\Big|\sum_{i\in I} \lambda_i \sum_{s \in J} a_s^i f_s(t)\Big| \leq \max\Big\{ \Big|\sum_{i\in E} \lambda_i u_i(t)\Big| : \ & E \subset I,\ (u_i)_{i\in E} \text{ is} \\ & \mathcal{G}^+ - \text{admissible}\Big\} \\ & + \Big(\sum_{i=1}^{\infty} \delta_i\Big) \max_{i\in I} \lambda_i + \sum_{i\in I} \lambda_i \sum_{s\notin J} a_s^i |f_s(t)|.\end{aligned}$$

Note that when $(u_i)_{i\in E}$ is $\mathcal{G}^+$-admissible, we have

$$\Big\|\sum_{i\in E} \lambda_i u_i\Big\| \leq \big(p(C+1)+1\big) \max_{i\in E} \lambda_i < \big(p(C+1)+1\big)\delta.$$

Hence,

$$\Big|\sum_{i\in I} \lambda_i \sum_{s\in J} a_s^i f_s(t)\Big| < \big(p(C+1)+2\big)\delta + \sum_{i\in I} \lambda_i \sum_{s\notin J} a_s^i |f_s(t)|.$$

Next, put $I^c = \{1, \ldots, n\} \setminus I$. Then,

$$\sum_{i\in I^c} \lambda_i \sum_s a_s^i < \delta, \text{ as } \Big\|\sum_{i\in I^c} \lambda_i u_i\Big\|_{\ell_1} < \delta.$$

Combining the preceding estimates we conclude

$$\begin{aligned}\Big|\sum_{i=1}^{n} \lambda_i \sum_{s\in J} a_s^i f_s(t)\Big| &\leq \sum_{i\in I^c} \lambda_i \sum_s a_s^i + \Big|\sum_{i\in I} \lambda_i \sum_{s\in J} a_s^i f_s(t)\Big| \\ &< \delta + \big(p(C+1)+2\big)\delta + \sum_{i\in I} \lambda_i \sum_{s\notin J} a_s^i |f_s(t)| \\ &< \epsilon + \sum_{i=1}^{n} \lambda_i \sum_{s\notin J} a_s^i |f_s(t)|.\end{aligned}$$

Therefore, u is (τ, p, ϵ)-large. This completes the proof. □

We are now ready for the

Proof of Theorem 6.7. We claim that every infinite subset of N contains a further infinite subset which is α-nice. If this claim holds, then evidently, N is itself α-nice. So suppose on the contrary, that the claim is false and

choose $N_0 \in [N]$ having no infinite subset which is α-nice. We now claim that there exist $1 \leq \beta_1 < \alpha$ and $N_1 \in [N_0]$ which is β_1-nice. Indeed, define

$$\beta_1 = \min\{\zeta < \omega_1 : \exists\, M \in [N_0] \text{ such that no block basis of } 0-\text{averages supported by } M \text{ is a } c_0^\zeta-\text{spreading model}\}.$$

Since N is α-large, α belongs to the set and so $1 \leq \beta_1 \leq \alpha$. Choose $N_1 \in [N_0]$ such that no block basis of 0-averages supported by N_1 is a $c_0^{\beta_1}$-spreading model. We show N_1 is β_1-nice. Because N_0 is assumed to contain no infinite subset which is α-nice, we shall also obtain $\beta_1 < \alpha$.

Let $M \in [N_1]$, $\beta < \beta_1$, $p \in \mathbb{N}$ and $\epsilon > 0$. We shall find a β_1-average supported by M which is (β, p, ϵ)-large. Since $\beta < \beta_1$, there exist $M_1 \in [M]$ and a constant $C > 0$ such that the block basis $(f_m)_{m \in M_1}$ is a c_0^β-spreading model with constant $C > 0$. Let $0 < \delta < \epsilon/(pC)$. Since no block basis of 0-averages supported by M_1 is a $c_0^{\beta_1}$-spreading model, Theorem 5.1 yields a β_1-average u, supported by M_1, positive scalars $(\lambda_i)_{i \in F}$ (where $F = \operatorname{supp} u$) and $I \subset F$ with $I \in S_{\beta_1}$, such that

$$u = \sum_{i \in F} \lambda_i f_i,\ \max_{i \in I} \lambda_i < \delta, \text{ and } \sum_{i \in F \setminus I} \lambda_i < \delta.$$

Let $t \in K$ and let J be the union of less than, or equal to, p consecutive members of S_β. It follows that

$$\begin{aligned} \Big|\sum_{i \in J \cap F} \lambda_i f_i(t)\Big| &\leq \Big\|\sum_{i \in J \cap F} \lambda_i f_i\Big\| \\ &\leq pC \max_{i \in F} \lambda_i < pC\delta < \epsilon. \end{aligned}$$

Thus, u is a β_1-average, (β, p, ϵ)-large, and so N_1 is β_1-nice, as claimed.

We shall now construct, by transfinite induction on $1 \leq \tau < \omega_1$, families $\{N_\tau\}_{1 \leq \tau < \omega_1} \subset [N_0]$ and $\{\beta_\tau\}_{1 \leq \tau < \omega_1} \subset [1, \alpha)$ with the following properties:

(1) $N_{\tau_2} \setminus N_{\tau_1}$ is finite, for all $1 \leq \tau_1 < \tau_2 < \omega_1$.
(2) N_τ is β_τ-nice, for all $1 \leq \tau < \omega_1$.
(3) $\beta_{\tau_1} < \beta_{\tau_2}$, for all $1 \leq \tau_1 < \tau_2 < \omega_1$.

Of course, (3) is absurd since $\alpha < \omega_1$. Hence, our assumption that N_0 contained no infinite subset which is α-nice, was false. The proof of the theorem will be completed, once we give the construction of the above described families, satisfying conditions (1)-(3). N_1 and β_1 have been already constructed. Suppose that $1 < \tau_0 < \omega_1$ and that $\{N_\tau\}_{1 \leq \tau < \tau_0} \subset [N_0]$, $\{\beta_\tau\}_{1 \leq \tau < \tau_0} \subset [1, \alpha)$ have been constructed fulfilling properties (1)-(3), above, with ω_1 being replaced by τ_0.

Assume first that τ_0 is a successor ordinal, say $\tau_0 = \tau_1 + 1$. We know by the inductive construction, that N_{τ_1} is β_{τ_1}-nice. By assumption, N is α-large. Since $\beta_{\tau_1} < \alpha$, there exists $\Gamma \in [N_{\tau_1}]$ such that no block basis of β_{τ_1}-averages supported by Γ is a $c_0^{\eta_{\tau_1}}$-spreading model, where $\beta_{\tau_1} + \eta_{\tau_1} = \alpha$. Lemma 6.14 now implies the existence of $N_{\tau_0} \in [N_{\tau_1}]$ and $1 \leq \zeta_{\tau_1} \leq \eta_{\tau_1}$

such that N_{τ_0} is $(\beta_{\tau_1}+\zeta_{\tau_1})$-nice. Set $\beta_{\tau_0}=\beta_{\tau_1}+\zeta_{\tau_1}$. Necessarily, $\beta_{\tau_0}<\alpha$, by the choice of N_0. It is easy to see that the families $\{N_\tau\}_{1\leq\tau<\tau_0+1}$ and $\{\beta_\tau\}_{1\leq\tau<\tau_0+1}$ satisfy conditions (1)-(3), above, with ω_1 being replaced by τ_0+1.

Next assume that τ_0 is a limit ordinal and choose a strictly increasing sequence of ordinals $\tau_1<\tau_2<\dots$ such that $\tau_0=\lim_n\tau_n$. By the inductive construction we have that $\beta_{\tau_1}<\beta_{\tau_2}<\dots$ and thus we may define the limit ordinal $\beta_{\tau_0}=\lim_n\beta_{\tau_n}$. In addition to this, $N_{\tau_n}\setminus N_{\tau_m}$ is finite for all integers $m<n$. We deduce from the above, that $\cap_{i=1}^k N_{\tau_i}$ is β_{τ_k}-nice, for all $k\in\mathbb{N}$. Finally, choose $N_{\tau_0}\in[N_0]$ such that $N_{\tau_0}\setminus\cap_{i=1}^k N_{\tau_i}$ is finite, for all $k\in\mathbb{N}$. We infer from Lemma 6.12, that N_{τ_0} is β_{τ_0}-nice. It is easily verified now, that the families $\{N_\tau\}_{1\leq\tau<\tau_0+1}$ and $\{\beta_\tau\}_{1\leq\tau<\tau_0+1}$ satisfy conditions (1)-(3), above, with ω_1 being replaced by τ_0+1. This completes the inductive step and the proof of the theorem. $\square$

Proof of Corollary 6.3. Assume without loss of generality, that (f_n) has no subsequence equivalent to the unit vector basis of c_0. By the Kunen-Martin boundedness principle (see [16], [25]), we may choose an ordinal $1\leq\gamma<\omega_1$ such that no subsequence of (f_n) is a c_0^γ-spreading model. Set $K_m=\{t\in K:\sum_n|f_n(t)|\leq m\}$, for all $m\in\mathbb{N}$. Clearly, (K_m) is an increasing sequence of closed subsets of K and $K=\cup_m K_m$. We claim that for every $m\in\mathbb{N}$, every $N\in[\mathbb{N}]$, and all $\epsilon>0$, there exists a γ-average u of (f_n) supported by N and such that $|u|(t)<\epsilon$, for all $t\in K_m$ (if $u=\sum_i a_i f_i$, we define $|u|(x)=\sum_i|a_i||f_i(x)|$, for all $x\in K$).

To see this, let $0<\delta<\epsilon/m$. Since no subsequence of (f_n) is a c_0^γ-spreading model, Theorem 5.1 allows us choose a γ-average u of (f_n), supported by N and such that there exist non-negative scalars $(\lambda_i)_{i=1}^p$ and $I\subset\{1,\dots,p\}$ satisfying the following: (1) $u=\sum_{i=1}^p\lambda_i f_i$ and $\max_{i\in I}\lambda_i<\delta$. (2) $(f_i)_{i\in I}$ is S_γ-admissible (i.e. $I\in S_\gamma$) and $\sum_{i\in\{1,\dots,p\}\setminus I}\lambda_i<\delta$. It is easy to check now that for every $t\in K_m$ we have $|u|(t)<\epsilon$ and thus our claim holds.

Now let (ϵ_n) be a summable sequence of positive scalars and $N\in[\mathbb{N}]$. Successive applications of the previous claim yield a block basis $v_1<v_2<\dots$ of γ-averages of (f_n), supported by N and satisfying $|v_n|(t)<\epsilon_n$ for every $t\in K_n$ and all $n\in\mathbb{N}$. It follows that for all $t\in K$ the set $\{n\in\mathbb{N}:|v_n(t)|\geq\epsilon_n\}$ is a subset of $\{1,\dots,q_t\}$, where q_t is the least $m\in\mathbb{N}$ such that $t\in K_m$. We deduce from Theorem 6.1, that there exist $\beta<\omega_1$ and a block basis of β-averages of (v_n), equivalent to the unit vector basis of c_0.

In order to get a block basis of averages of (f_i) equivalent to the unit vector basis of c_0, one needs a somewhat more demanding argument which goes as follows. Choose a countable limit ordinal α with $\gamma<\alpha$ and let $(\alpha_j+1)_{j=1}^\infty$ be the sequence of ordinals associated to α. Let $N\in[\mathbb{N}]$ and choose $n\in N$ with $n\geq 2$, such that $\gamma<\alpha_n$. Let $m\in\mathbb{N}$. Since no subsequence of (f_i) is a $c_0^{\alpha_n}$-spreading model, our preceding argument allows us choose an α_n-average v of (f_i), supported by $\{i\in N:\,n<i\}$, and such that $|v|(t)<1/(2n)$, for all

$t \in K_m$. Set $u = \big((1/n)f_n + v\big)/\|(1/n)f_n + v\|$. Clearly, u is an α-average of (f_i) supported by N and satisfying $|u|(t) < 3/n$, for all $t \in K_m$. Note that $n = \min \operatorname{supp} u$.

Summarizing, given $N \in [\mathbb{N}]$ we can select a block basis $u_1 < u_2 < \dots$ of α-averages of (f_i) supported by N and satisfying $|u_n|(t) < 3/m_n$, for all $t \in K_n$ and $n \in \mathbb{N}$. In the above, we have let $m_n = \min \operatorname{supp} u_n$, for all $n \in \mathbb{N}$. It follows that for all $n \in \mathbb{N}$, if $t \in K_n$ and $|u_i|(t) \geq 3/m_i$, then $i < n$. Given $L \in [\mathbb{N}]$, set $l_n = \min \operatorname{supp} \alpha_n^L$, for all $n \in \mathbb{N}$. We now define

$$\mathcal{D} = \{L \in [\mathbb{N}] : \forall\, n \in \mathbb{N}, \forall\, t \in K_n, \text{ if } |\alpha_i^L|(t) \geq 3/l_i, \text{ then } i < n\}.$$

$\mathcal{D}$ is closed in the topology of pointwise convergence, thanks to Lemma 4.3. Our preceding discussion and Lemma 4.3, show that every $N \in [\mathbb{N}]$ contains some $L \in \mathcal{D}$ as a subset. We infer from Theorem 4.5, that $[N] \subset \mathcal{D}$ for some $N \in [\mathbb{N}]$.

Next, let $\mathcal{T}_0$ be the collection of those finite subsets E of N that can be written in the form $E = \cup_{i=1}^m \operatorname{supp} \alpha_i^L$, for some $L \in [N]$ (depending on E) for which there exists some $t \in K$ (depending on E and L) such that $|\alpha_i^L|(t) \geq 3/l_i$, for all $i \leq m$.

Let $\mathcal{T}$ be the collection of all initial segments of elements of $\mathcal{T}_0$. We claim that $\mathcal{T}$ is compact in the topology of pointwise convergence. Indeed, were this false, there would exist $M \in [N]$, $M = (m_i)$, such that $\{m_1, \dots, m_n\} \in \mathcal{T}$, for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. It follows that $\cup_{i=1}^n \operatorname{supp} \alpha_i^M \in \mathcal{T}$. Hence, there exist $L_n \in [N]$, $k_n \in \mathbb{N}$ and $t_n \in K$ such that $\cup_{i=1}^n \operatorname{supp} \alpha_i^M$ is an initial segment of $\cup_{i=1}^{k_n} \operatorname{supp} \alpha_i^{L_n}$ and $|\alpha_i^{L_n}|(t_n) \geq 3/d_i$, for all $i \leq k_n$, where $d_i = \min \operatorname{supp} \alpha_i^{L_n}$, for all $i \in \mathbb{N}$. We now deduce from Lemma 4.3, that $n \leq k_n$ and that $\alpha_i^M = \alpha_i^{L_n}$, for all $i \leq n$. Therefore, $|\alpha_i^M|(t_n) \geq 3/m_i$, for all $i \leq n$, where $m_i = \min \operatorname{supp} \alpha_i^M$, for all $i \in \mathbb{N}$. The compactness of K now implies that there is some $t \in K$ satisfying $|\alpha_i^M|(t) \geq 3/m_i$, for all $i \in \mathbb{N}$. This is a contradiction, as $M \in \mathcal{D}$. Thus, our claim holds and so $\mathcal{T}$ is indeed compact.

We next apply a result from [31] to obtain $P \in [N]$ such that $\mathcal{T}[P]$ is a hereditary and compact family. The result in [21] now yields $Q \in [P]$ and a countable ordinal $\eta > \alpha$, such that $\mathcal{T}[Q] \subset S_\eta$. It follows that for every $L \in [Q]$ and all $n \in \mathbb{N}$ such that there exists some $t \in K$ satisfying $|\alpha_i^L|(t) \geq 3/l_i$, for all $i \leq n$, we have $\cup_{i=1}^n \operatorname{supp} \alpha_i^L \in S_\eta$.

We now claim that $\xi^Q \leq \eta$ (see Definition 6.4). If this is not the case, we may choose $R \in [Q]$, $R = (r_i)$, which is ζ-large, for some countable ordinal ζ with $\eta < \zeta$. Let $\epsilon > 0$. We shall assume, as we clearly may, that $\sum_i (1/r_i) < \epsilon$. Since $\alpha < \eta$, we may choose an ordinal β with $\alpha + \beta = \zeta$. By passing to an infinite subset of R, if necessary, we may assume without loss of generality, thanks to Proposition 5.9, that every ζ-average of (f_i) supported by R admits an $(\epsilon, \alpha, \beta)$-decomposition.

Because R is ζ-large, it is also ζ-nice, by Theorem 6.7. We may thus select a ζ-average u of (f_i), supported by R, which is $(\eta, 1, \epsilon)$-large. We infer from

Proposition 5.9 that there exist normalized blocks $u_1 < \cdots < u_n$, positive scalars $(\lambda_i)_{i=1}^n$ and $I \subset \{1, \dots, n\}$ such that $u = \sum_{i=1}^n \lambda_i u_i$ and u_i is an α-average for all $i \in I$, while $\| \sum_{i \notin I} \lambda_i u_i \|_{\ell_1} < \epsilon$.

Now let $t \in K$ and define $H = \{i \in I : |u_i|(t) \geq 3/q_i\}$, where $q_i = \min \operatorname{supp} u_i$, for all $i \in I$. Let $\{i_1 < \dots, < i_k\}$ be an enumeration of H. Lemma 4.3 yields some $L \in [R]$ such that $u_{i_j} = \alpha_j^L$, for all $j \leq k$. Set $J = \cup_{i \in H} \operatorname{supp} u_i$. Since $L \in [Q]$, it follows that $J \in S_\eta$. Writing $u_i = \sum_s a_s^i f_s$, for all $i \leq n$, we conclude, as u is $(\eta, 1, \epsilon)$-large, that

$$\Big|\sum_{i \in H} \lambda_i \sum_s a_s^i f_s(t)\Big| \leq \epsilon + \sum_{i \notin H} \lambda_i \sum_s a_s^i |f_s(t)|.$$

We now have the estimates

$$\begin{aligned}
|u(t)| &= \Big|\sum_{i \in H} \lambda_i \sum_s a_s^i f_s(t) + \sum_{i \notin H} \lambda_i \sum_s a_s^i f_s(t)\Big| \\
&\leq \Big|\sum_{i \in H} \lambda_i \sum_s a_s^i f_s(t)\Big| + \Big|\sum_{i \notin H} \lambda_i \sum_s a_s^i f_s(t)\Big| \\
&\leq \epsilon + 2 \sum_{i \notin H} \lambda_i \sum_s a_s^i |f_s(t)| \\
&\leq \epsilon + 2 \sum_{i \in I \setminus H} \lambda_i \sum_s a_s^i |f_s(t)| + 2 \sum_{i \notin I} \lambda_i \sum_s a_s^i |f_s(t)| \\
&\leq \epsilon + 6 \sum_{i \in I \setminus H} |u_i|(t) + 2 \Big\|\sum_{i \notin I} \lambda_i u_i\Big\|_{\ell_1} \\
&< \epsilon + 18 \sum_{i \in I \setminus H} (1/q_i) + 2\epsilon \\
&< 21\epsilon.
\end{aligned}$$

Since $\|u\| = 1$, we reach a contradiction for ϵ small enough. Therefore, $\xi^Q \leq \eta$. Proposition 6.5 now yields a block basis of ξ-averages of (f_i), for some $\xi \leq \eta$, equivalent to the unit vector basis of c_0. $\square$

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